Composite Invariants and Unoriented Topological String Amplitudes

Chandrima Paul[1], Pravina Borhade [2], P. Ramadevi[3]
Department of Physics,
Indian Institute of Technology Bombay,
Mumbai 400 076, India

Abstract

Sinha and Vafa [1] had conjectured that the SO Chern-Simons gauge theory on S^3 must be dual to the closed A-model topological string on the orientifold of a resolved conifold. Though the Chern-Simons free energy could be rewritten in terms of the topological string amplitudes providing evidence for the conjecture, we needed a novel idea in the context of Wilson loop observables to extract cross-cap $c = 0, 1, 2$ topological amplitudes. Recent paper of Marino [2] based on the work of Morton and Ryder [3] has clearly shown that the composite representation placed on the knots and links plays a crucial role to rewrite the topological string cross-cap $c = 0$ amplitude. This enables extracting the unoriented cross-cap $c = 2$ topological amplitude. In this paper, we have explicitly worked out the composite invariants for some framed knots and links carrying composite representations in $U(N)$ Chern-Simons theory. We have verified generalised Rudolph's theorem, which relates composite invariants to the invariants in $SO(N)$ Chern-Simons theory, and also verified Marino's conjectures on the integrality properties of the topological string amplitudes. For some framed knots and links, we have tabulated the BPS integer invariants for cross-cap $c = 0$, $c = 1$ and $c = 2$ giving the open-string topological amplitude on the orientifold of the resolved conifold.

[1]E-mail:chandrima@phy.iitb.ac.in
[2]E-mail: borhadepravina@gmail.com
[3]Email: ramadevi@phy.iitb.ac.in

1 Introduction

We have seen interesting developments in the open string and closed string dualities during the last 12 years starting from the celebrated work of Maldacena [4]. Gopakumar and Vafa [5–7] conjectured open-closed duality in the topological string context. Gopakumar-Vafa conjecture states that the A-model open topological string theory on the deformed conifold, equivalent to the Chern-Simons gauge theory on S^3 [8], is dual to the closed string theory on a resolved conifold.

In ref. [5], it was shown that the free-energy expansion of $U(N)$ Chern-Simons field theory on S^3 at large N resembles A-model topological string theory amplitudes on the resolved conifold. This provided an evidence for the conjecture. Another piece of evidence at the level of observables was shown by Ooguri and Vafa [9] for the simplest Wilson loop observable (simple circle also called unknot) in Chern-Simons theory on S^3. In particular, Ooguri-Vafa considered the expectation value of a scalar operator $\mathcal{Z}_{\mathcal{H}}(v)$ in the topological string theory corresponding to the simple circle in submanifold S^3 of the deformed conifold and showed its form in the resolved conifold background. From these results for unknot, Ooguri-Vafa conjectured on the form for $\mathcal{Z}_{\mathcal{H}}(v)$ for any knot or link in S^3. For completeness and simplicity, we briefly present the form for knots:

$$\mathcal{F}_{\mathcal{H}}(v) = \ln \mathcal{Z}_{\mathcal{H}}(v) = \ln\{\sum_R \mathcal{H}_R[\mathcal{K}] s_R(v)\} \; = \sum_{R,d} f_R(q^d, \lambda^d) s_R(v^d) \tag{1.1}$$

$$\text{where } f_R(q, \lambda) = \frac{1}{(q^{1/2} - q^{-1/2})} \sum_{Q,s} N_{R,Q,s} \lambda^Q q^s \tag{1.2}$$

Here $\mathcal{H}_R(\mathcal{K})$ are the $U(N)$ Chern-Simons invariants for a knot $\mathcal{K}$ in S^3 carrying representation R and $s_R(v)$ are the Schur polynomials in variable v which represent $U(N)$ holonomy of the knot $\mathcal{K}$ in the Lagragian submanifold $\mathcal{N}$ which intersects S^3 along the knot. $\mathcal{F}_{\mathcal{H}}(v)$ denotes the free-energy of the topological open-string partition function on the resolved conifold and $f_R(q, \lambda)$ are the $U(N)$ reformulated invariants. The conjecture states that the reformulated invariant must have the form (1.2) where $N_{R,Q,s}$ are integer coefficients.

Labastida-Marino [10] used group-theoretic techniques to rewrite the expectation value of the topological operators in terms of link invariants in $U(N)$ Chern-Simons field theory on S^3. This group theoretic approach enabled verification of Ooguri-Vafa conjecture for many non-trivial knots [10–13]. Conversely, the Ooguri-Vafa conjecture led to a reformulation of Chern-Simons field theory invariants for knots and links giving new polynomial invariants(1.2). The integer coefficients of these new polynomial invariants have topological meaning accounting for BPS states in the string theory. The challenge still remains in obtaining such integers for non-trivial knots and links within topological string theory.

Another challenging question is to attempt similar duality conjectures between Chern-Simons gauge theories on three-manifolds other than S^3 and closed string theories. Invoking Gopakumar-Vafa conjecture and Ooguri-Vafa conjecture, it was possible to explicitly write the $U(N)$ Chern-Simons free-energy expansion at large N as a closed string theoretic expansion [14]. Surprisingly, the expansion resembled partition function of a closed string theory on a Calabi-Yau background with one kahler parameter. Unfortunately, the Chern-Simons free-energy expansion for other three-manifolds are not equivalent to the 't Hooft large N perturbative expansion around a classical solution [15]. In order to predict new duality conjectures, we need to extract the perturbative expansion around a classical solution from the free-energy.

For orbifolds of S^3, which gives Lens space $\mathcal{L}[p,1] \equiv S^3/Z_p$, it is believed that the Chern-Simons theory is dual to the A-model closed string theory on A_{p-1} fibred over P^1 Calabi-Yau background. It was Marino [16] who showed that the perturbative Chern-Simons theory on Lens space $\mathcal{L}[p,1]$ can be given a matrix model description. Also, hermitian matrix model description of B-model topological strings [17] was shown to be equivalent to Marino's matrix model using mirror symmetry [18]. It is still a challenging open problem to look for dual closed string description corresponding to $U(N)$ Chern-Simons theory on other three-manifolds.

The extension of these duality conjectures for other gauge groups like $SO(N)$ and $Sp(N)$ have also been studied. In particular, the free-energy expansion $F^{(SO)}_{(CS)}[S^3]$ of the Chern-Simons theory on S^3 based on SO gauge group was shown to be dual to A-model closed string theory on a orientifold of the resolved conifold background [1]. In particular, the string partition function Z for these orientifolding action must have two contributions:

$$F^{(SO)}_{(CS)}[S^3] \; = Z \; = \; \frac{1}{2} Z^{or} + Z^{(unor)} \tag{1.3}$$

where $Z^{(or)}$ is the untwisted contribution and $Z^{(unor)}$ is the twisted sector contribution. The untwisted contribution exactly matches the $U(N)$ Chern-Simons free energy on S^3. Using the topological vertex as a tool, Bouchard et al [19, 20] have determined unoriented closed string amplitude and unoriented open topological string amplitudes for a few orientifold toric geometry with or without D-branes.

In Ref. [21], the generalisation of Ooguri-Vafa conjecture for observables involving $SO(N)$ holonomy, different from the works of Bouchard et al [19,20], was studied. Similar to the $U(N)$ result (1.2), the coefficients of $SO(N)$ reformulated invariants are indeed integers.

Following Sinha-Vafa conjecture [1], the expectation value of the topological string operator (observables) $Z_{\mathcal{G}}(v)$ where $\mathcal{G}$ represents $SO(N)$ knot invariants in Chern-Simons theory on S^3 and v represents the SO holonomy on the submanifold $\mathcal{N}$ intersecting S^3 along a knot. It is expected that the free-energy of the open-string partition function on the orientifold of the

resolved conifold must also satisfy a relation similar to eqn.(1.3):

$$\mathcal{F}_{\mathcal{G}}(v) = \ln Z_{\mathcal{G}}(v) = \frac{1}{2}\mathcal{F}^{(or)}_{\mathcal{R}}(v) + \mathcal{F}^{(unor)}(v) \ . \tag{1.4}$$

where $\mathcal{F}^{(or)}_{\mathcal{R}}(v)$ is the oriented or untwisted sector contribution (also called cross-cap $c = 0$) and the twisted sector term $\mathcal{F}^{(unor)}(v)$ will have both cross-cap $c = 1$ and $c = 2$ contributions to the open topological string amplitudes. It was not clear [19,20] as to how to obtain $\mathcal{F}^{(or)}_{\mathcal{R}}(v)$ in the orientifold theory using $U(N)$ Chern-Simons knot invariants. As a result, it was not possible to distinguish the topological amplitudes of cross-cap $c = 0$ from $c = 2$ contribution. However using parity argument in variable $\sqrt{\lambda}$, the cross-cap $c = 1$ topological amplitudes contribution could be obtained [19–21].

From the orientifolding action, Marino [2] has indicated that there must be a $U(N)$ composite representation (R, S) placed on the knot in S^3 and the oriented contribution must be rewritable as:

$$\mathcal{F}^{(or)}_{\mathcal{R}}(v) = \sum_{R,S} \mathcal{H}_{(R,S)}[\mathcal{K}] s_R(v) s_S(v) = \sum_{R} \mathcal{R}_R[\mathcal{K}] s_R(v) \tag{1.5}$$

where $s_R(v)$ and $s_S(v)$ are the Schur polynomials corresponding to the $U(N)$ holonomy in two Lagrangian submanifolds $\mathcal{N}_\epsilon$ and $\mathcal{N}_{-\epsilon}$ related by the orientifolding action. Here ϵ denotes the deformation parameter of the deformed conifold. The oriented invariant $\mathcal{R}_R[\mathcal{K}]$ can be obtained from composite invariants $\mathcal{H}_{(R,S)}[\mathcal{K}]$ using the properties of the Schur polynomials. Though we have so far discussed for knots, it is straightforward to generalise these arguments for any r-component link L.

In this paper, we explicitly evaluate the composite invariants $\mathcal{H}_{(R_1,S_1),(R_2,S_2),\ldots(R_r,S_r)}[L]$, in $U(N)$ Chern-Simons gauge theory for many framed knots and links L made of r component knots $\mathcal{K}_\alpha$'s carrying composite representations (R_α, S_α) using the tools [22]. These composite invariants are polynomials in two variables q, λ. We find that the framing factor for the component knots of the links carrying composite representation requires a slightly modified choice of the $U(1)$ charge so that the composite invariants are polynomials in variables q and λ.

Comparing these invariants with $SO(N)$ Chern-Simons invariants $\mathcal{G}_{R_1,R_2,\ldots R_r}[L]$ [21] for link L whose components carry representations R_α's which are also polynomials in two variables (q, λ), we have verified the generalised Rudolph's theorem [3,23]:

$$\frac{1}{2}\left[\mathcal{H}_{(R,R)}[\mathcal{K}] + \{\mathcal{G}_R[\mathcal{K}]\}^2\right] = f(q)\sum_{n,p} a_{n,p}\lambda^{\frac{n}{2}} q^p \ , \tag{1.6}$$

for many framed knots $\mathcal{K}$ carrying $R = \begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square, \square$. Here $f(q)$ is a function of q, $a_{n,p}$ are integers. In fact, the above relation between $U(N)$ composite invariants and $SO(N)$ invariants appears naturally from the integrality properties of the topological string amplitudes in the orientifold

geometry [2]. Using these composite representation invariants, we verified the integrality conjectures of Marino [2] for framed knots and framed two-component links. While submitting this paper, we came across a recent paper [24] where Marino's conjectures have been verified for standard framing torus knots and torus links which is a special case of our results.

The organisation of the paper is as follows. In section 2, we present composite framed knot and framed two-component link invariants in $U(N)$ Chern-Simons theory. In section 3, we briefly review Marino's conjectures on the reformulated invariants of the framed links in the orientifold resolved conifold. In section 4, we verify one of Marino's conjectures and tabulate the $c = 0$ BPS integer coefficients for few examples. In section 5, we obtain the reformulated invariants corresponding to the unoriented topological string amplitude. Indeed, these reformulated invariants also obey the integrality conjecture of Marino. We have tabulated the $c = 1$ and $c = 2$ BPS integers for some framed knots and framed Hopf link in section 6. In the concluding section, we summarize the results obtained. In appendix A, we present $U(N)$ composite invariants for some framed knots and framed two-component links for some representations. In appendix B, the unoriented reformulated polynomial invariants for few non-trivial framed knots and framed links are presented.

2 Chern-Simons Gauge theory and Composite Link invariants

Chern-Simons gauge theory on S^3 based on the gauge group G is described by the following action:

$$S = \frac{k}{4\pi} \int_{S^3} Tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \tag{2.1}$$

where A is a gauge connection for compact semi-simple gauge group G and k is the coupling constant. The observables in this theory are Wilson loop operators:

$$W_{R_1, R_2, \ldots R_r}[L] \ = \ \prod_{\alpha=1}^{r} Tr_{R_\alpha} U[\mathcal{K}_\alpha] \ , \tag{2.2}$$

where $U[\mathcal{K}_\alpha] = P \left[\exp \oint_{\mathcal{K}_\alpha} A \right]$ denotes the holonomy of the gauge field A around the component knot $\mathcal{K}_\alpha$ of a r-component link L carrying representation R_α. The expectation value of these Wilson loop operators are the link invariants:

$$\langle W_{R_1, R_2, \ldots R_r}[L] \rangle (q, \lambda) = \frac{\int [\mathcal{D}A] e^{iS} W_{R_1, R_2, \ldots, R_r}[L]}{\int [\mathcal{D}A] e^{iS}} \ , \tag{2.3}$$

These link invariants are polynomials in two variables

$$q = \exp \left(\frac{2\pi i}{k + C_v} \right) \ , \ \lambda = q^{N+a} \ , \tag{2.4}$$

where C_v is the dual coxeter number of the gauge group G

$$C_v = \begin{cases} N & \text{for } G = SU(N) \\ N-2 & \text{for } G = SO(N) \end{cases} \text{ and } a = \begin{cases} 0 & \text{for } G = SU(N) \\ -1 & \text{for } G = SO(N) \end{cases}$$

These link invariants can be computed using the following two inputs [22]:
(i) Any link can be drawn as a closure or plat of braids,
(ii) The connection between Chern-Simons theory and the Wess-Zumino conformal field theory.
We now define some quantities which will be useful later. The quantum dimension of a representation R with highest weight Λ_R is given by

$$dim_q R = \Pi_{\alpha>0} \frac{[\alpha \cdot (\rho + \Lambda_R)]}{[\alpha \cdot \rho]} \ , \tag{2.5}$$

where α's are the positive roots and ρ is the Weyl vector equal to the sum of the fundamental weights of the group G. The square bracket refers to the quantum number defined by

$$[x] = \frac{\left(q^{x/2} - q^{-x/2}\right)}{(q^{1/2} - q^{-1/2})} \ . \tag{2.6}$$

The $SU(N)$ quadratic Casimir for representation R is given by

$$C_R = -\frac{\ell^2}{2N} + \kappa_R = -\frac{\ell^2}{2N} + \frac{1}{2}\left((N+a)\ell + \ell + \sum_i (l_i^2 - 2il_i)\right) \ . \tag{2.7}$$

Our interest is to obtain invariants of framed knots and framed links carrying representation $R_c \equiv (R, S)$ called composite representation in $U(N)$ Chern-Simons gauge theory so that Marino's conjectures on the topological amplitudes in the orientifold of resolved conifold geometry can be verified.

2.1 Composite Invariants in $U(N)$ Chern-Simons Gauge Theory

The composite representation, $R_c \equiv (R, S)$ labelled by a pair of Young diagram is defined as [2, 25–27]

$$R_c \equiv (R, S) = \sum_{U,V,W} (-1)^{l(U)} N^R_{UV} N^S_{U^T W} (V \times \bar{W}) \ , \tag{2.8}$$

where U, V, W are the representations of the group U(N) , $\ell(U)$ denotes the number of boxes in the Young diagram corresponding to U and N is the Littlewood-Richardson coefficient for multiplication of the Young diagrams.

If we take the simplest defining representation for $R = \square$ and $S = \square$, then the composite representation $R_c = (\square, \square)$ derived from eqn. (2.8) will be the adjoint representation of $U(N)$.

In terms of fundamental weights, the highest weight of R_c is $\Lambda^{(1)} + \Lambda^{(N-1)}$. Using the above eqn.(2.8), one can obtain the $SU(N)$ representation for any composite representation (R, S) and the corresponding highest weight will be $\Lambda_R + \Lambda_{\bar{S}}$ where Λ_R and $\Lambda_{\bar{S}}$ are the highest weights of representation R and conjugate representation $\bar{S}$ respectively.

We will now explicitly evaluate the polynomials for various knots and links carrying the composite representation (R, S) in $U(N)$ Chern-Simons theory. For the simplest circle called unknot U_p with an arbitrary framing p, the composite invariant will be framing factor multiplying the quantum dimension of the composite representation (R, S):

$$\mathcal{H}_{(R,S)}[U_p] = (-1)^{\ell p} q^{\frac{p\{n_{(R,S)}\}^2}{2}} q^{pC_{(R,S)}} dim_q(R, S) \ . \tag{2.9}$$

where ℓ is the total number of boxes in the Young diagram for composite representation (R, S), $C_{(R,S)}$ denotes the $SU(N)$ quadratic casimir (2.7) and $n_{(R,S)}$ represents the $U(1)$ charge for the composite representation (R, S). From the definition of the composite representation highest weight, it appears that the $U(1)$ charge $n_{(R,S)}$ must be must be difference of $U(1)$ charges n_R and n_S of representation R and S:

$$n_{(R,S)} = |n_R - n_S| \ . \tag{2.10}$$

The $U(1)$ charges were chosen [13, 14] such the $U(N)$ invariants are polynomials in two variables q, λ [13, 14]. For representation R with $\ell(R)$ number of boxes in the Young diagram representation, the $U(1)$ charge n_R is

$$n_R = \frac{\ell(R)}{\sqrt{N}} \ . \tag{2.11}$$

Substituting the $U(1)$ charge (2.11) in eqn.(2.10), the unknot invariant (2.9) simplies to

$$\mathcal{H}_{(R,S)}[U_p] = (-1)^{\ell p} q^{\kappa_R + \kappa_S} dim_q(R, S) \ . \tag{2.12}$$

In other words, the framing factor for the knots carrying composite representation (R, S) involves only the sum of κ_R and κ_S as defined in eqn. (2.7). Now, we can write the $U(N)$ framed knot invariants for torus knots of the type $(2, 2m + 1)$ with framing p as follows:

$$\mathcal{H}_{(R,S)}[\mathcal{K}](q, \lambda) = (-1)^{\ell p} q^{p(\kappa_R + \kappa_S)} \sum_{R_t} dim_q R_t (\lambda_t)^{2m+1} \ , \tag{2.13}$$

where $R_t \in (R, S) \otimes (R, S)$ and λ_t is the braiding eigenvalue in standard framing ($p = 0$) for parallely oriented strands:

$$\lambda_t = \epsilon_t q^{2C_{(R,S)} - C_{R_t}/2} \ , \tag{2.14}$$

where $\epsilon_{R_t} = \pm 1$ depending upon whether the representation R_t appears symmetrically or antisymmetrically with respect to the tensor product $(R,S) \otimes (R,S)$ in the $U(N)_k$ Wess-Zumino Witten model. Unlike the totally symmetric or totally antisymmetric representations, the tensor product of composite representations does give multiplicities and we fix the sign of ϵ_{R_t} by imposing isotopy equivalence of two or more knots. In fact, fixing the sign of the eigenvalues for such composite representation was a non-trivial exercise. So, in appendix A, we have explicitly given all the irreducible representations R_t and the signs ϵ_t for some composite representations so that the composite invariants can be computed.

The $U(N)$ invariants for framed torus links of the type $(2, 2m)$ can also be written. For example, the $U(N)$ invariant for a Hopf link of type $(2, 2)$ with linking number -1 and framing numbers p_1 and p_2 on the component knots carrying representations (R_1, S_1) and (R_2, S_2) will be

$$\mathcal{H}_{(R_1,S_1),(R_2,S_2)}[H](q,\lambda) = (-1)^{\ell_1 p_1 + \ell_2 p_2} q^{p_1(\kappa_{R_1}+\kappa_{S_1}) + p_2(\kappa_{R_2}+\kappa_{S_2})} \times \\ q^{\ell k n_{(R_1,S_1)} n_{(R_2,S_2)}} \sum_{R_t} dim_q R_t q^{C_{(R_1,S_1)} + C_{(R_2,S_2)} - C_{R_t}} , \qquad (2.15)$$

where $\ell k = -1$ is the linking number between the two-components and $R_t \in (R_1, S_1) \otimes (R_2, S_2)$. We now explicitly evaluate the knot polynomials carrying the composite representation $(\square, \square)$ in $U(N)$ Chern-Simons theory, for the knots upto five crossings. For the simplest composite representation $(\square, \square)$, which we denote by ρ_0, the highest weight is $\Lambda^{(N-1)} + \Lambda^{(1)}$. The p-frame unknot invariant for this representation is

$$\mathcal{H}_{(\square,\square)}[U_p] = (-1)^{\ell p} \lambda^p (dim_q \rho_0) = (-1)^{Np} \lambda^p [N-1][N+1] , \qquad (2.16)$$

where rewriting the quantum numbers (2.6) will give the p-framed unknot invariant in variables $q, \lambda = q^N$. The highest weights for all the representations R_t's obtained from $\rho_0 \otimes \rho_0$ and their corresponding quantum dimensions(2.5) with the braiding eigenvalues (2.14) are tabulated below:

R_t	Highest weight	Quantum Dimension	λ_t
R_1	$\Lambda^{(N)} + \Lambda^{(N-2)} + 2\Lambda^{(1)}$	$dim_q R_1 = \frac{[N-1][N-2][N+1][N+2]}{[2][2]}$	$\lambda_1 = -\lambda$
R_2	$2\Lambda^{(N-1)} + \Lambda^{(2)}$	$dim_q R_2 = \frac{[N-1][N-2][N+1][N+2]}{[2][2]}$	$\lambda_2 = -\lambda$
R_3	$\Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(2)}$	$dim_q R_3 = \frac{[N]^2[N-3][N+1]}{[2][2]}$	$\lambda_3 = q\lambda$
R_4	$\Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(1)}$	$dim_q R_4 = [N-1][N+1];$	$\lambda_4 = \lambda^{3/2}$

R_t	Highest weight	Quantum Dimension	λ_t
R_5	$2\Lambda^{(N)}$	$dim_q R_5 = 1$	$\lambda_5 = \lambda^2$
R_6	$2\Lambda^{(N-1)} + 2\Lambda^{(1)}$	$dim_q R_6 = \frac{[N]^2[N+3][N-1]}{[2][2]}$	$\lambda_6 = q^{-1}\lambda$
R_7	$\Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(1)}$	$dim_q R_7 = [N-1][N+1]$	$\lambda_7 = -\lambda^{3/2}$

Substituting the tabulated data in eqn.(2.13), the knot invariants for the framed trefoil and 5-crossing knots with framing p carrying representation $\rho_0 = (\square, \square)$ which have unknot invariants as an overall factor:

$$\begin{aligned}\mathcal{H}_{(\square,\square)}[\mathcal{K}_3] = \mathcal{H}_{\square,\square}[U_0]\Big(&\lambda^2\lambda^p(q^{-2}+q^2+2) - \lambda^3(2q^{-2}+q^{-1}+q-2q^2-2)\\ &+\lambda^4(q^{-2}-2q^{-1}-2q+q^2+3)+\lambda^5(q^{-1}+q-2)\Big)\end{aligned} \tag{2.17}$$

$$\begin{aligned}\mathcal{H}_{(\square,\square)}[\mathcal{K}_5] = \mathcal{H}_{\square,\square}[U_0]\Big[&\lambda^4\lambda^p(q^{-4}+2q^{-2}+2q^2+q^4+3)\\ &-\lambda^5(2q^{-4}+q^{-3}-4q^{-2}+q^{-1}+q-4q^2+q^3-2q^4-4)\\ &+\lambda^6(q^{-4}-2q^{-3}+3q^{-2}-2q^{-1}-2q+3q^2-2q^3+q^4+4)\\ &+\lambda^7(q^{-3}-2q^{-2}+2q^{-1}+2q-2q^2+q^3-2)\\ &+\lambda^8(q^{-2}-2q^{-1}-2q+q^2+2)+\lambda^9(q^{-1}+q-2)\Big]\end{aligned} \tag{2.18}$$

We have presented the tensor products and knot invariants for other composite representations in the appendix A. These composite invariants play a very crucial role in obtaining the topological string amplitudes corresponding to cross-caps $c = 0, 1$ and 2. Using these $U(N)$ composite invariants and the $SO(N)$ invariants in appendix A of Ref. [21], we have verified generalised Rudolph's theorem (1.6).

3 Reformulated Link Invariants

We will now review the conjectures proposed by Marino [2] for the reformulated $SO(N)$ invariants of knots and links. Particularly, we have to get the untwisted sector (oriented) contribution (1.4)to the open topological string amplitudes on the orientifold of the resolved conifold geometry.

Using the properties satisfied by Schur polynomials, eqn.(1.5) implies that the oriented invariants $\mathcal{R}_{R_1,\ldots,R_r}[L]$ of the link L whose components $\mathcal{K}_1,\ldots,\mathcal{K}_r$ are colored by representations $R_1,\ldots,R_r$ is given by

$$\mathcal{R}_{R_1,\ldots,R_r}[L] = \sum_{S_1,T_1,\ldots,S_r,T_r}\prod_{\alpha=1}^{r} N^{R_\alpha}_{S_\alpha,T_\alpha}\mathcal{H}_{(S_1,T_1),\ldots,(S_r,T_r)}[L]\ , \tag{3.1}$$

where $N^{R_\alpha}_{S_\alpha,T_\alpha}$ are the Littlewood-Richardson coefficients and $\mathcal{H}_{(S_1,T_1),\ldots,(S_r,T_r)}[L]$ are composite invariants in $U(N)$ Chern-Simons gauge theory of the link whose components carry the composite representations $(S_1,T_1),\ldots,(S_r,T_r)$ of $U(N)$. The generating functional giving the oriented contribution to the open topological string partition function (1.4) is defined as

$$\mathcal{Z}_{\mathcal{R}}(v_1,\ldots,v_r) = \sum_{R_1,\ldots,R_r} \mathcal{R}_{R_1,\ldots,R_r}[L] \prod_{\alpha=1}^{r} s_{R_\alpha}(v_\alpha);\ \mathcal{F}_{\mathcal{R}}(v_1,\ldots,v_r) = \log \mathcal{Z}_{\mathcal{R}}(v_1,\ldots,v_r)\ , \quad (3.2)$$

where $s_R(v)$ are the Schur polynomials. Also the generating functionals for those involving $SO(N)$ Chern-Simons invariants , $\mathcal{G}_{R_1,\ldots,R_r}$, of a link L are defined as

$$\mathcal{Z}_{\mathcal{G}}(v_1,\ldots,v_r) = \sum_{R_1,\ldots,R_r} \mathcal{G}_{R_1,\ldots,R_r}[L] \prod_{\alpha=1}^{r} s_{R_\alpha}(v_\alpha);\ \mathcal{F}_{\mathcal{G}}(v_1,\ldots,v_r) = \log \mathcal{Z}_{\mathcal{G}}(v_1,\ldots,v_r)\ . \quad (3.3)$$

Marino [2] has conjectured a specific form for these generating functionals:

$$\mathcal{F}_{\mathcal{R}}(v_1,\ldots,v_r) = \sum_{d=1}^{\infty} \sum_{R_1,\ldots,R_r} h_{R_1,\ldots,R_r}(q^d,\lambda^d) \prod_{\alpha=1}^{r} s_{R_\alpha}(v_\alpha^d)\ , \quad (3.4)$$

and

$$\mathcal{F}_{\mathcal{G}}(v_1,\ldots,v_r) - \frac{1}{2}\mathcal{F}_{\mathcal{R}}(v_1,\ldots,v_r) = \sum_{d\,odd} \sum_{R_1,\ldots,R_r} g_{R_1,\ldots,R_r}(q^d,\lambda^d) \prod_{\alpha=1}^{r} s_{R_\alpha}(v_\alpha^d)\ , \quad (3.5)$$

where $h_{R_1,\ldots,R_r}(q,\lambda)$ and $g_{R_1,\ldots,R_r}(q,\lambda)$ are the reformulated polynomial invariants involving the $U(N)$ and $SO(N)$ Chern-Simons link invariants respectively. The reformulated invariants are polynomials in q and λ and conjectured to obey the following form

$$h_{R_1,\ldots,R_r}(q,\lambda)\ \text{or}\ g_{R_1,\ldots,R_r}(q,\lambda) = \sum_{Q,s} \frac{1}{q^{1/2}-q^{-1/2}} \tilde{N}_{R_1,\ldots,R_r,Q,s} q^s \lambda^Q\ , \quad (3.6)$$

where $\tilde{N}_{R_1,\ldots,R_r,Q,s}$ are integers. Though we know that the reformulated invariants $f_R(q,\lambda)$ obtained from $U(N)$ invariants $\mathcal{H}_R[L]$ satisfies the conjecture (1.2), it is not at all obvious that the reformulated invariant $h_{R_1,\ldots,R_r}(q,\lambda)$ corresponding to the oriented invariants (3.1) involving linear combination of $U(N)$ composite invariants must obey a similar conjectured form (3.6). We check few examples in section 4 to verify Marino's conjecture on the oriented reformulated invariants. These reformulated invariants are further refined using the following equations, in order to reveal the BPS structure

$$h_{R_1,\ldots,R_r}(q,\lambda) = \sum_{S_1,\ldots,S_r} M_{R_1,\ldots,R_r;S_1,\ldots,S_r}\, \hat{h}_{S_1,\ldots,S_r}(q,\lambda), \quad (3.7)$$

$$g_{R_1,\ldots,R_r}(q,\lambda) = \sum_{S_1,\ldots,S_r} M_{R_1,\ldots,R_r;S_1,\ldots,S_r}\, \hat{g}_{S_1,\ldots,S_r}(q,\lambda)\ , \quad (3.8)$$

where

$$M_{R_1,\dots R_r;S_1,\dots S_r} = \sum_{T_1,\dots T_r} \prod_{\alpha=1}^{r} C_{R_\alpha S_\alpha T_\alpha} S_{T_\alpha}(q) \ , \tag{3.9}$$

$R_\alpha, S_\alpha, T_\alpha$ are representations of the symmetric group S_{ℓ_α} which can be labelled by a Young-Tableau with a total of ℓ_α boxes and C_{RST} are the Clebsch-Gordan coefficients of the symmetric group. $S_R(q)$ is non-zero only for the hook representations. For a hook representation having $\ell - d$ boxes in the first row of Young tableau with total ℓ boxes, $S_R(q) = (-1)^d q^{-(\ell-1)/2+d}$. Marino [2] has conjectured that the refined reformulated invariants $\hat{h}_{R_1,\dots,R_r}(q,\lambda)$ and $\hat{g}_{R_1,\dots,R_r}(q,\lambda)$ should have the following structure:

$$\hat{h}_{R_1,\dots,R_r}(q,\lambda) = z^{r-2} \sum_{g\geq 0} \sum_{Q} N^{c=0}_{R_1,\dots,R_r,g,Q} z^{2g} \lambda^Q \ , \tag{3.10}$$

$$\hat{g}_{R_1,\dots,R_r}(q,\lambda) = z^{r-1} \sum_{g\geq 0} \sum_{Q} \left(N^{c=1}_{R_1,\dots,R_r,g,Q} z^{2g} \lambda^Q + N^{c=2}_{R_1,\dots,R_r,g,Q} z^{2g+1} \lambda^Q \right) \ , \tag{3.11}$$

where $N^{c=0}_{R_1,\dots,R_r,g,Q}$, $N^{c=1}_{R_1,\dots,R_r,g,Q}$ and $N^{c=2}_{R_1,\dots,R_r,g,Q}$ are the BPS invariants corresponding to cross-caps $c = 0, 1$ and 2 respectively and the variable $z = q^{1/2} - q^{-1/2}$.

In the next three sections, we obtain the reformulated invariants and obtain the BPS integers coefficients for framed knots and framed two-component links.

4 Computation of Oriented Invariants $h_{R_1,\dots,R_r}(q,\lambda)$ and BPS invariants $N^{c=0}_{R_1,\dots R_r,g,Q}$

In this section, we list the reformulated oriented invariants and the corresponding BPS invariants for simple framed knots like unknot and trefoil knot to verify the conjecture (3.6).

4.1 Framed unknot

The reformulated invariants (3.6) corresponding to the oriented invariants for the unknot with framing p are given below

$$h_{\square} = \frac{1}{(-1+q)\sqrt{\lambda}} \left(2\,(-1)^p\, \lambda^{p/2} \sqrt{q}\ (-1+\lambda) \right) \tag{4.1}$$

$$h_{\square\square} = \frac{-\lambda^{p-1}}{(-1+q)^2\ (1+q)} \Big(\lambda - 2\, q^{1+p}\ (-1+\lambda)\ (-1+q\,\lambda) + (-1)^p\ (-1+q)\ q \left(-1+\lambda^2\right) \\ +q \left(1+q+(-3+(-3+q)\ q)\ \lambda + (1+q)\ \lambda^2 \right) \Big) \tag{4.2}$$

$$h_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{\lambda^{p-1}}{\left(-1+q\right)^2\,\left(1+q\right)} \Big(-2\,q^{1-p}\,\left(q-\lambda\right)\,\left(-1+\lambda\right) + \left(-1\right)^p\,\left(-1+q\right)\,q\,\left(-1+\lambda^2\right)$$
$$- \left(1+q\right)\,\left(\lambda + q\,\left(1+\lambda\,\left(-4+q+\lambda\right)\right)\right)\Big) \qquad (4.3)$$

These results alongwith eqs.(3.7) and (3.10) give the following BPS invariants.

Unknot with framing $p = 0$

$$N^{c=0}_{\square,0,\pm 1/2} = \pm 2, \qquad N^{c=0}_{\square\square,0,0} = 1, \qquad N^{c=0}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},0,0} = 1.$$

Unknot with framing $p = 1$

g	Q=0	1
0	2	-2

$N^{c=0}_{\square,g,Q}$

$$N^{c=0}_{\square\square,0,1} = 1$$

g	Q=1	2
0	3	-2

$N^{c=0}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},g,Q}$

Unknot with framing $p = 2$

g	Q=1/2	3/2
0	-2	2

$N^{c=0}_{\square,g,Q}$

g	Q=2	3
0	3	-2

$N^{c=0}_{\square\square,g,Q}$

g	Q=1	2	3
0	-2	7	-4
1	0	2	-2

$N^{c=0}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},g,Q}$

4.2 Framed trefoil knot

For the trefoil knot with framing p, the oriented invariants are

$$h_{\square} = \frac{1}{\left(-1+q\right)\,\sqrt{q}} \left(2\left(-1\right)^p\,\lambda^{p/2}\left(-1+\lambda\right)\,\sqrt{\lambda}\,\left(1+q^2-q\,\lambda\right)\right) \qquad (4.4)$$

$$h_{\square\square} = \frac{\lambda^{p+1}}{\left(-1+q\right)^2 q^2\,\left(1+q\right)} \Big(-2\,q^{1+p}\,\left(-1+\lambda\right) + 2\,q^{2+p}\,\left(-1+\lambda\right)\,\lambda + 2\,q^{3+p}\,\left(-1+\lambda\right)\,\lambda$$
$$-\left(-1+\lambda\right)^2\lambda - q^7\left(-1+\lambda\right)^2\lambda - 2\,q^{8+p}\left(-1+\lambda\right)^2\lambda + q\,\left(1+\left(-1\right)^p-\lambda\right)\,\left(-1+\lambda^2\right)$$
$$-q^6\,\left(-1+\left(-1\right)^p+\lambda\right)\left(-1+\lambda^2\right) + 2\,q^{7+p}\left(-1+\lambda\right)^2\left(1+\lambda^2\right) - 2\,q^{6+p}\,\lambda\,\left(2-3\,\lambda+\lambda^2\right)$$
$$-2\,q^{4+p}\,\left(-1+2\,\lambda-2\,\lambda^2+\lambda^3\right) - 2\,q^{5+p}\,\left(-1+2\,\lambda-2\,\lambda^2+\lambda^3\right)$$
$$+q^5\,\left(-1-\left(-1\right)^p+4\,\lambda+\left(-7+\left(-1\right)^p\right)\,\lambda^2+2\,\lambda^3+\lambda^5\right)$$
$$+q^2\,\left(-1+\left(-1\right)^p+4\,\lambda-\left(7+\left(-1\right)^p\right)\,\lambda^2+2\,\lambda^3+\lambda^5\right)$$
$$-q^4\,\left(2-8\,\lambda+\left(9+\left(-1\right)^p\right)\,\lambda^2-6\,\lambda^3-\left(-1+\left(-1\right)^p\right)\,\lambda^4+\lambda^5\right)$$
$$-q^3\,\left(2-8\,\lambda-\left(-9+\left(-1\right)^p\right)\,\lambda^2-6\,\lambda^3+\left(1+\left(-1\right)^p\right)\,\lambda^4+\lambda^5\right)\Big) \qquad (4.5)$$

$$
\begin{aligned}
h_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = {} & \frac{\lambda^{p+1}}{(-1+q)^2\, q^3\, (1+q)} \Big(-2\, q^{7-p}\, (-1+\lambda) + 2\, q^{5-p}\, (-1+\lambda)\, \lambda + 2\, q^{6-p}\, (-1+\lambda)\, \lambda \\
& -2\, q^{-p}(-1+\lambda)^2\, \lambda - q\, (-1+\lambda)^2\, \lambda - q^8\, (-1+\lambda)^2\, \lambda + q^7\, (1+(-1)^p-\lambda)\, \left(-1+\lambda^2\right) \\
& -q^2\, (-1+(-1)^p+\lambda)\, \left(-1+\lambda^2\right) + 2\, q^{1-p}\, (-1+\lambda)^2\, \left(1+\lambda^2\right) - 2\, q^{2-p}\, \lambda\, \left(2-3\,\lambda+\lambda^2\right) \\
& -2\, q^{3-p}\, \left(-1+2\,\lambda-2\,\lambda^2+\lambda^3\right) - 2\, q^{4-p}\, \left(-1+2\,\lambda-2\,\lambda^2+\lambda^3\right) \\
& +q^3\, \left(-1-(-1)^p+4\,\lambda+(-7+(-1)^p)\,\lambda^2+2\,\lambda^3+\lambda^5\right) \\
& +q^6\, \left(-1+(-1)^p+4\,\lambda-(7+(-1)^p)\,\lambda^2+2\,\lambda^3+\lambda^5\right) \\
& -q^4\, \left(2-8\,\lambda+(9+(-1)^p)\,\lambda^2-6\,\lambda^3-(-1+(-1)^p)\,\lambda^4+\lambda^5\right) \\
& -q^5\, \left(2-8\,\lambda-(-9+(-1)^p)\,\lambda^2-6\,\lambda^3+(1+(-1)^p)\,\lambda^4+\lambda^5\right)\Big)
\end{aligned} \tag{4.6}
$$

From eqns.(3.7) and (3.10) we obtain the BPS invariants corresponding to cross-cap $c=0$.

Trefoil knot with framing $p=0$

g	Q=1/2	3/2	5/2
0	-4	6	-2
1	-2	2	0

$N^{c=0}_{\square,g,Q}$

g	Q=1	2	3	4	5	6
0	-4	25	-40	25	-4	-1
1	-2	18	-32	18	-2	0
2	0	3	-6	3	0	0

$N^{c=0}_{\square\!\square,g,Q}$

g	Q=1	2	3	4	5	6
0	-8	41	-64	41	-8	-1
1	-8	46	-76	46	-8	0
2	-2	17	-30	17	-2	0
3	0	2	-4	2	0	0

$N^{c=0}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},g,Q}$

Trefoil knot with framing $p=1$

g	Q=1	2	3
0	4	-6	2
1	2	-2	0

$N^{c=0}_{\square,g,Q}$

g	Q= 2	3	4	5	6	7
0	-8	41	-64	41	-8	-1
1	-8	46	-76	46	-8	0
2	-2	17	-30	17	-2	0
3	0	2	-4	2	0	0

g	Q=2	3	4	5	6	7
0	-14	69	-104	63	-12	-1
1	-22	116	-180	108	-22	0
2	-12	73	-120	71	-12	0
3	-2	20	-36	20	-2	0
4	0	2	-4	2	0	0

$N^{c=0}_{\square\square,g,Q}$ $N^{c=0}_{\substack{\square\\\square},g,Q}$

Trefoil knot with framing $p=2$

g	Q=3/2	5/2	7/2
0	-4	6	-2
1	-2	2	0

$N^{c=0}_{\square,g,Q}$

g	Q=3	4	5	6	7	8
0	-14	69	-104	63	-12	-1
1	-22	116	-180	108	-22	0
2	-12	73	-120	71	-12	0
3	-2	20	-36	20	-2	0
4	0	2	-4	2	0	0

$N^{c=0}_{\square\square,g,Q}$

g	Q=3	4	5	6	7	8
0	-26	109	-154	91	-18	-1
1	-58	256	-376	226	-48	0
2	-46	241	-378	227	-44	0
3	-16	110	-186	108	-16	0
4	-2	24	-44	24	-2	0
5	0	2	-4	2	0	0

$N^{c=0}_{\substack{\square\\\square},g,Q}$

5 Explicit Computation of $SO(N)$ reformulated invariants $g_{R_1,R_2,\ldots R_r}(q,\lambda)$

In this section we compute the functions $g_{R_1,\ldots R_r}(q,\lambda)$ for various nontrivial framed knots and links and show that they obey the conjectured form (3.6).

5.1 Framed Unknot

$$g_{\square} = (-1)^p\,\lambda^{p/2} \tag{5.1}$$

$$g_{\square\square} = \frac{1}{(-1+q)}\lambda^{p-1/2}\left(-\sqrt{q}+q^{\frac{1}{2}+p}\right)(-1+\lambda) \tag{5.2}$$

$$g_{\substack{\square\\\square}} = -\frac{q^{-p}\,\lambda^{p-1/2}}{(-1+q)}\left(-\sqrt{q}+q^{\frac{1}{2}+p}\right)(-1+\lambda) \tag{5.3}$$

$$\begin{aligned} g_{\square\square\square} = {} & \frac{\lambda^{3p/2-1}}{(-1+q)^2\,(1+q)}\left((-1)^p\,q\,(-1+q^p)\,(-1+\lambda)\right. \\ & \left.\left(1+q-q^p-q^{2\,p}-\lambda-q\,\lambda+q^{1+p}\,\lambda+q^{1+2\,p}\,\lambda\right)\right) \end{aligned} \tag{5.4}$$

$$
\begin{aligned}
g_{\yng(2,1)} = & \frac{-\lambda^{3p/2-1}}{(-1+q)^2\,(1+q)}(-1)^p\,q^{1-p}\,(-1+q^p)\,(-1+\lambda)\,(1-q\,(-2+\lambda) \\
& +q^p\,(-2+\lambda)-2\,\lambda+q^{1+p}\,(-1+2\,\lambda)\Big)
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
g_{\yng(1,1,1)} = & \frac{\lambda^{3p/2-1}}{(-1+q)^2\,(1+q)}(-1)^p\,q^{\frac{1}{2}-3\,p}\,(-1+\lambda)\,\Big(-q^{\frac{3}{2}}+q^{\frac{1}{2}+2\,p}\,(1-2\,\lambda) \\
& -q^{\frac{3}{2}+2\,p}\,(-2+\lambda)+q^{\frac{1}{2}+3\,p}\,(-1+\lambda)+q^{\frac{3}{2}+3\,p}\,(-1+\lambda)+\sqrt{q}\,\lambda\Big)
\end{aligned}
\tag{5.6}
$$

5.2 Framed Trefoil Knot

$$
g_{\yng(1)} = (-1)^p\,q^{-1}\,\lambda^{p/2+1}\,\Big(1-q^2\,(-1+\lambda)-\lambda+q\,\Big(1-\lambda+\lambda^2\Big)\Big)
\tag{5.7}
$$

$$
\begin{aligned}
g_{\yng(2)} = & \frac{\lambda^p}{(-1+q)\,q^2}\,(-1+\lambda)\,\lambda^{\frac{3}{2}}\,\Big(q^{\frac{1}{2}+p}+q^{\frac{9}{2}+p}\,(1-2\,\lambda)+\sqrt{q}\,(-1+\lambda) \\
& +q^{\frac{9}{2}}\,(-1+\lambda)-q^{\frac{5}{2}+p}\,(-1+\lambda)-q^{\frac{3}{2}+p}\,\lambda+q^{\frac{15}{2}+p}\,(-1+\lambda)\,\lambda \\
& +q^{\frac{19}{2}+p}\,\lambda^2-q\,\lambda^{\frac{5}{2}}+q^4\,\lambda^{\frac{5}{2}}+q^{7+p}\,\lambda^{\frac{5}{2}}-q^{10+p}\,\lambda^{\frac{5}{2}}+q^{\frac{3}{2}}\,\Big(-1+2\,\lambda-2\,\lambda^2\Big) \\
& +q^{\frac{7}{2}}\,\Big(-1+2\,\lambda-2\,\lambda^2\Big)-q^{\frac{17}{2}+p}\,\lambda\,\Big(1+\lambda^2\Big)+q^{\frac{7}{2}+p}\,\Big(1-\lambda+\lambda^2\Big) \\
& +q^{\frac{11}{2}+p}\,\Big(1-\lambda+\lambda^2\Big)+q^{\frac{13}{2}+p}\,\Big(1-\lambda+\lambda^2\Big)+q^2\,\lambda^{\frac{3}{2}}\,\Big(1+\lambda+\lambda^2\Big) \\
& -q^3\,\lambda^{\frac{3}{2}}\,\Big(1+\lambda+\lambda^2\Big)-q^{8+p}\,\lambda^{\frac{3}{2}}\,\Big(1+\lambda+\lambda^2\Big)+q^{9+p}\,\lambda^{\frac{3}{2}}\,\Big(1+\lambda+\lambda^2\Big) \\
& +q^{\frac{5}{2}}\,\Big(-2+3\,\lambda-\lambda^2+\lambda^3\Big)\Big)
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
g_{\yng(1,1)} = & \frac{\lambda^p}{-1+q}\,q^{-7-p}\,(-1+\lambda)\,\lambda^{\frac{3}{2}}\,\Big(q^{\frac{19}{2}}+q^{\frac{11}{2}}\,(1-2\,\lambda)-q^{\frac{15}{2}}\,(-1+\lambda) \\
& +q^{\frac{11}{2}+p}\,(-1+\lambda)+q^{\frac{19}{2}+p}\,(-1+\lambda)-q^{\frac{17}{2}}\,\lambda+q^{\frac{5}{2}}\,(-1+\lambda)\,\lambda+\sqrt{q}\,\lambda^2 \\
& +\lambda^{\frac{5}{2}}-q^3\,\lambda^{\frac{5}{2}}-q^{6+p}\,\lambda^{\frac{5}{2}}+q^{9+p}\,\lambda^{\frac{5}{2}}+q^{\frac{13}{2}+p}\,\Big(-1+2\,\lambda-2\,\lambda^2\Big) \\
& +q^{\frac{17}{2}+p}\,\Big(-1+2\,\lambda-2\,\lambda^2\Big)+q^{\frac{7}{2}}\,\Big(1-\lambda+\lambda^2\Big)+q^{\frac{9}{2}}\,\Big(1-\lambda+\lambda^2\Big) \\
& +q^{\frac{13}{2}}\,\Big(1-\lambda+\lambda^2\Big)-q\,\lambda^{\frac{3}{2}}\,\Big(1+\lambda+\lambda^2\Big)+q^2\,\lambda^{\frac{3}{2}}\,\Big(1+\lambda+\lambda^2\Big) \\
& +q^{7+p}\,\lambda^{\frac{3}{2}}\,\Big(1+\lambda+\lambda^2\Big)-q^{8+p}\,\lambda^{\frac{3}{2}}\,\Big(1+\lambda+\lambda^2\Big)-q^{\frac{3}{2}}\,\Big(\lambda+\lambda^3\Big) \\
& +q^{\frac{15}{2}+p}\,\Big(-2+3\,\lambda-\lambda^2+\lambda^3\Big)\Big)
\end{aligned}
\tag{5.9}
$$

Substituting values for p, the above equations reduce to the conjectured form (3.6). We have presented the reformulated invariants for few framed knots and two component links in appendix B.

6 $N^{c=1}_{(R_1,\ldots R_r),g,Q}$ and $N^{c=2}_{(R_1,\ldots R_r),g,Q}$ Computation

We shall now compute the integer coefficients corresponding to cross-cap $c = 1$ and $c = 2$ unoriented open string amplitude obtained from $SO(N)$ reformulated invariants for various framed knots and framed links using eqns.(3.8, 3.11).

6.1 Framed Knots

For unknot with zero framing, the only non zero coefficient is $N^{c=1}_{\square,0,0} = 1$.

Unknot with framing $p = 1$

$$N^{c=1}_{\square,0,1/2} = -1\,. \tag{6.1}$$

g	Q=1/2	3/2
0	1	-1

$N^{c=1}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},g,Q}$ for the unknot

Unknot with framing $p = 2$

$$N^{c=1}_{\square,0,1} = 1\,. \tag{6.2}$$

g	Q=3/2	5/2
0	1	-1

$N^{c=1}_{\square\square,g,Q}$ for the unknot

g	Q=3/2	5/2
0	3	-3
1	1	-1

$N^{c=1}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},g,Q}$ for the unknot

Trefoil knot with framing $p = 1$

g	Q=5/2	7/2	9/2	11/2	13/2
0	16	-69	111	-79	21
1	20	-146	307	-251	70
2	8	-128	366	-330	84
3	1	-56	230	-220	45
4	0	-12	79	-78	11
5	0	-1	14	-14	1
6	0	0	1	-1	0

$N^{c=1}_{\square\square,g,Q}$ for the trefoil knot

g	Q= 4	5	6	7
0	21	-63	63	-21
1	70	-231	231	-70
2	84	-322	322	-84
3	45	-219	219	-45
4	11	-78	79	-11
5	1	-14	14	-1
6	0	-1	1	0

$N^{c=2}_{\square\square,g,Q}$ for the trefoil knot

g	Q= 5/2	7/2	9/2	11/2	13/2
0	30	-114	167	-111	28
1	55	-311	587	-457	126
2	36	-367	912	-791	210
3	10	-230	770	-715	165
4	1	-79	376	-364	66
5	0	-14	106	-105	13
6	0	-1	16	-16	1
7	0	0	1	-1	0

$N^{c=1}_{\square,g,Q}$ for the trefoil knot

g	Q= 4	5	6	7
0	28	-84	84	-28
1	126	-406	406	-126
2	210	-756	756	-210
3	165	-705	705	-165
4	66	-363	363	-66
5	13	-105	105	-13
6	1	-16	16	-1
7	0	-1	1	0

$N^{c=2}_{\square,g,Q}$ for the trefoil knot

Trefoil knot with framing $p = 2$

g	Q= 7/2	9/2	11/2	13/2	15/2
0	30	-114	167	-111	28
1	55	-311	587	-457	126
2	36	-367	912	-791	210
3	10	-230	770	-715	165
4	1	-79	376	-364	66
5	0	-14	106	-105	13
6	0	-1	16	-16	1
7	0	0	1	-1	0

$N^{c=1}_{\square\square,g,Q}$ for the trefoil knot

g	Q=5	6	7	8
0	28	-84	84	-28
1	126	-406	406	-126
2	210	-756	756	-210
3	165	-705	705	-165
4	66	-363	363	-66
5	13	-105	105	-13
6	1	-16	16	-1
7	0	-1	1	0

$N^{c=2}_{\square\square,g,Q}$ for the trefoil knot

g	Q= 7/2	9/2	11/2	13/2	15/2
0	50	-174	237	-149	36
1	125	-601	1042	-776	210
2	120	-919	2046	-1709	462
3	55	-771	2222	-2001	495
4	12	-376	1443	-1365	286
5	1	-106	574	-560	91
6	0	-16	137	-136	15
7	0	-1	18	-18	1
8	0	0	1	-1	0

$N^{c=1}_{\square,g,Q}$ for the trefoil knot

g	Q= 5	6	7	8
0	36	-108	108	-36
1	210	-666	666	-210
2	462	-1596	1596	-462
3	495	-1947	1947	-495
4	286	-1353	1353	-286
5	91	-559	559	-91
6	15	-136	136	-15
7	1	-18	18	-1
8	0	-1	1	0

$N^{c=2}_{\square,g,Q}$ for the trefoil knot

Torus knot $(2, 5)$ with framing $p = 0$

g	Q=1	2	3
0	3	-3	1
1	1	-1	0

$N^{c=1}_{\square,g,Q}$

g	Q=7/2	9/2	11/2	13/2	15/2	17/2
0	80	-285	285	55	-225	90
1	260	-1190	1190	910	-1875	705
2	336	-2192	2192	3801	-6315	2178
3	221	-2286	2286	7666	-11385	3498
4	78	-1456	1456	8997	-12364	3289
5	14	-575	575	6642	-8567	1911
6	1	-137	137	3180	-3876	695
7	0	-18	18	986	-1140	154
8	0	-1	1	191	-210	19
9	0	0	0	21	-22	1
10	0	0	0	1	-1	0

$N^{c=1}_{\square\square,g,Q}$ for torus knot $(2,5)$

g	Q=5	7	8	10
0	45	-225	225	-45
1	330	-1875	1875	-330
2	924	-6315	6315	-924
3	1287	-11385	11385	-1287
4	1001	-12364	12364	-1001
5	455	-8567	8567	-455
6	120	-3876	3876	-120
7	17	-1140	1140	-17
8	1	-210	210	-1
9	0	-22	22	0
10	0	-1	1	0

$N^{c=2}_{\square\square,g,Q}$ for torus knot $(2,5)$

These results are agreeing with the results in Ref. [24].

g	Q=7/2	9/2	11/2	13/2	15/2	17/2
0	120	-415	415	45	-275	110
1	490	-2085	2085	1215	2750	1045
2	819	-4663	4663	6364	-11110	3927
3	724	-5994	5994	15644	-24090	7722
4	365	-4822	4822	22372	-31746	9009
5	105	-2500	2500	20370	-27118	6643
6	16	-833	833	12307	-15503	3180
7	1	-172	172	4998	-5985	986
8	0	-20	20	1349	-1540	191
9	0	-1	1	232	-253	21
10	0	0	0	23	-24	1
11	0	0	0	1	-1	0

$N^{c=1}_{\square,g,Q}$ for torus knot $(2,5)$

g	Q=5	7	8	10
0	55	-275	275	-55
1	495	-2750	2750	-495
2	1716	-11110	11110	-1716
3	3003	-24090	24090	-3003
4	3003	-31746	31746	-3003
5	1820	-27118	27118	-1820
6	680	-15503	15503	-680
7	153	-5985	5985	-153
8	19	-1540	1540	-19
9	1	-253	253	-1
10	0	-24	24	0
11	0	-1	1	0

$N^{c=2}_{\square,g,Q}$ for torus knot $(2,5)$

Torus knot $(2,5)$ with framing $p=1$

g	Q=3/2	5/2	7/2
0	-3	3	-1
1	-1	1	0

$N^{c=1}_{\square,g,Q}$

g	Q= 9/2	11/2	13/2	15/2	17/2	19/2
0	120	-415	415	45	-275	110
1	490	-2085	2085	1215	-2750	1045
2	819	-4663	4663	6364	-11110	3927
3	724	-5994	5994	15644	-24090	7722
4	365	-4822	4822	22372	-31746	9009
5	105	-2500	2500	20370	-27118	6643
6	16	-833	833	12307	-15503	3180
7	1	-172	172	4998	-5985	986
8	0	-20	20	1349	-1540	191
9	0	-1	1	232	-253	21
10	0	0	0	23	-24	1
11	0	0	0	1	-1	0

$N^{c=1}_{\square\square,g,Q}$ for torus knot $(2,5)$

g	Q=6	8	9	11
0	55	-275	275	-55
1	495	-2750	2750	-495
2	1716	-11110	11110	-1716
3	3003	-24090	24090	-3003
4	3003	-31746	31746	-3003
5	1820	-27118	27118	-1820
6	680	-15503	15503	-680
7	153	-5985	5985	-153
8	19	-1540	1540	-19
9	1	-253	253	-1
10	0	-24	24	0
11	0	-1	1	0

$N^{c=2}_{\square\square,g,Q}$ for torus knot $(2,5)$

g	Q= 9/2	11/2	13/2	15/2	17/2	19/2
0	175	-585	580	28	-330	132
1	875	-3480	3460	1554	-3905	1496
2	1820	-9282	9261	10136	-18656	6721
3	2055	-14384	14376	29985	-47905	15873
4	1377	-14184	14183	51391	-75218	22451
5	561	-9247	9247	56470	-77415	20384
6	136	-4029	4029	41804	-54248	12308
7	18	-1159	1159	21317	-26333	4998
8	1	-211	211	7505	-8855	1349
9	0	-22	22	1792	-2024	232
10	0	-1	1	277	-300	23
11	0	0	0	25	-26	1
12	0	0	0	1	-1	0

$N^{c=1}_{\square\!\!\square,g,Q}$ for torus knot $(2,5)$

g	Q=6	8	9	11
0	66	-330	330	-66
1	715	-3905	3905	-715
2	3003	-18656	18656	-3003
3	6435	-47905	47905	-6435
4	8008	-75218	75218	-8008
5	6188	-77415	77415	-6188
6	3060	-54248	54248	-3060
7	969	-26333	26333	-969
8	190	-8855	8855	-190
9	21	-2024	2024	-21
10	1	-300	300	-1
11	0	-26	26	0
12	0	-1	1	0

$N^{c=2}_{\square\!\!\square,g,Q}$ for torus knot $(2,5)$

<u>Torus knot $(2,5)$ with framing $p=2$</u>

g	Q= 2	3	4
0	3	-3	1
1	1	-1	0

$N^{c=1}_{\square,g,Q}$

g	Q= 11/2	13/2	15/2	17/2	19/2	21/2
0	175	-585	580	28	-330	132
1	875	-3480	3460	1554	-3905	1496
2	1820	-9282	9261	10136	-18656	6721
3	2055	-14384	14376	29985	-47905	15873
4	1377	-14184	14183	51391	-75218	22451
5	561	-9247	9247	56470	-77415	20384
6	136	-4029	4029	41804	-54248	12308
7	18	-1159	1159	21317	-26333	4998
8	1	-211	211	7505	-8855	1349
9	0	-22	22	1792	-2024	232
10	0	-1	1	277	-300	23
11	0	0	0	25	-26	1
12	0	0	0	1	-1	0

$N^{c=1}_{\square\!\square,g,Q}$ for torus knot $(2,5)$

g	Q=7	9	10	12
0	66	-330	330	-66
1	715	-3905	3905	-715
2	3003	-18656	18656	-3003
3	6435	-47905	47905	-6435
4	8008	-75218	75218	-8008
5	6188	- 77415	77415	-6188
6	3060	-54248	54248	-3060
7	969	-26333	26333	-969
8	190	-8855	8855	-190
9	21	-2024	2024	-21
10	1	-300	300	-1
11	0	-26	26	0
12	0	-1	1	0

$N^{c=2}_{\square\!\square,g,Q}$ for torus knot $(2,5)$

g	Q= 11/2	13/2	15/2	17/2	19/2	21/2
0	245	-795	780	4	-390	156
1	1470	-5545	5480	1910	-5395	2080
2	3724	-17444	17361	15456	-30108	11011
3	5215	-32075	32030	54461	-90376	30745
4	4445	-37932	37921	110395	-166595	51766
5	2394	-30178	30177	143961	-202930	56576
6	817	-16472	16472	127771	-170408	41820
7	171	-6175	6175	79440	-100929	21318
8	20	-1561	1561	34978	-42503	7505
9	1	-254	254	10857	-12650	1792
10	0	-24	24	2323	-2600	277
11	0	-1	1	326	-28	25
12	0	0	0	27	-1	1
13	0	0	0	1	0	0

$N^{c=1}_{\square,g,Q}$ for torus knot $(2,5)$

g	Q=7	9	10	12
0	78	-390	390	-78
1	1001	-5395	5395	-1001
2	5005	-30108	30108	-5005
3	12870	-90376	90376	-12870
4	19448	-166595	166595	-19448
5	18564	-202930	202930	-18564
6	11628	-170408	170408	-11628
7	4845	-100929	100929	-4845
8	1330	-42503	42503	-1330
9	231	-12650	12650	-231
10	23	-2600	2600	-23
11	1	-351	351	-1
12	0	-28	28	0
13	0	-1	1	0

$N^{c=2}_{\square,g,Q}$ for torus knot $(2,5)$

Connected sum of trefoil and trefoil with framing $p = 0$

g	Q=2	3	4	5
0	8	-14	9	-2
1	6	-11	6	-1
2	1	-2	1	0

$N^{c=1}_{\square,g,Q}$ for trefoil # trefoil

g	Q=7/2	9/2	11/2	13/2	15/2	17/2	19/2	21/2
0	143	-831	1950	-2366	1561	-525	66	2
1	404	-3144	8854	-11819	7544	-1488	-596	245
2	464	-5419	19211	-28097	14046	6348	-9194	2641
3	277	-5379	25184	-40255	6296	44160	-41756	11473
4	90	-3292	21666	-38551	-18588	110890	-98450	26235
5	15	-1256	12654	-26241	-38613	159091	-141400	35750
6	1	-290	5048	-13093	-36589	147270	-133378	31031
7	0	-37	1352	-4787	-21053	92681	-85919	17763
8	0	-2	232	-1243	-7860	40544	-38455	6784
9	0	0	23	-215	-1917	12353	-11954	1710
10	0	0	1	-22	-295	2574	-2531	273
11	0	0	0	-1	-26	350	-348	25
12	0	0	0	0	-1	28	-28	1
13	0	0	0	0	0	1	-1	0

$N^{c=1}_{\square\square,g,Q}$ for trefoil # trefoil

g	Q=4	5	6	7	8	9	10	11
0	-46	627	-2210	3524	-2891	1190	-193	-1
1	-115	2857	-12709	23835	-22244	10068	-1517	-121
2	-114	5764	-32974	73721	-78050	37511	-4648	-1210
3	-54	6412	-48952	133320	-160443	79607	-5171	-4719
4	-12	4241	-45575	155369	-214257	107758	1914	-9438
5	-1	1707	-27770	122272	-195972	98868	11907	-11011
6	0	410	-11234	66279	-126105	63513	15145	-8008
7	0	54	-2987	24753	-57626	28938	10608	-3740
8	0	3	-501	6247	-18593	9314	4652	-1122
9	0	0	-48	1016	-4138	2070	1309	-209
10	0	0	-2	96	-604	302	230	-22
11	0	0	0	4	-52	26	23	-1
12	0	0	0	0	-2	1	1	0

$N^{c=2}_{\square\square,g,Q}$ for trefoil # trefoil

g	Q=7/2	9/2	11/2	13/2	15/2	17/2	19/2	21/2
0	227	-1237	2756	-3206	2045	-671	84	2
1	801	-5621	14872	-19187	12269	-2861	-564	291
2	1190	-11771	38341	-54346	29773	5595	-12485	3697
3	955	-14403	59796	-92245	27489	67819	-68353	18942
4	444	-11132	61614	-104212	-18925	211571	-190697	51337
5	119	-5578	43750	-83517	-79482	364896	-323843	83655
6	17	-1803	21761	-49317	-100043	404876	-363462	87971
7	1	-362	7561	-21758	-73516	307780	-281842	62136
8	0	-41	1795	-7097	-35330	165164	-154547	30056
9	0	-2	277	-1653	-11446	63250	-60402	9976
10	0	0	25	-258	-2483	17202	-16719	2233
11	0	0	1	-24	-346	3248	-3201	322
12	0	0	0	-1	-28	405	-403	27
13	0	0	0	0	-1	30	-30	1
14	0	0	0	0	0	1	-1	0

$N_{\begin{smallmatrix}\square\\\square\end{smallmatrix},g,Q^{c=1}}$ for trefoil # trefoil

g	Q= 4	5	6	7	8	9	10	11
0	-74	869	-2921	4526	-3631	1466	-234	-1
1	-234	4781	-20075	36309	-32999	14673	-2311	-144
2	-310	11808	-62281	132252	-135333	64116	-8536	-1716
3	-212	16343	-110933	280964	-323696	159083	-13541	-8008
4	-77	13779	-125127	386321	-503851	252187	-3927	-19305
5	-14	7338	-93938	362625	-540953	272586	19812	-27456
6	-1	2479	-48041	238667	-413588	208502	36734	-24752
7	0	515	-16793	111146	-228664	115027	33457	-14688
8	0	60	-3945	36403	-91650	45984	18962	-5814
9	0	3	-595	8191	-26359	13199	7081	-1520
10	0	0	-52	1204	-5298	2650	1748	-252
11	0	0	-2	104	-706	353	275	-24
12	0	0	0	4	-56	28	25	-1
13	0	0	0	0	-2	1	1	0

$N^{c=2}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},g,Q}$ for trefoil # trefoil

6.2 Framed Links

We take Hopf Link $H(p_1, p_2)$ with linking number -1 and framing on the two component knots as p_1 and p_2. The integers $N^{c=1}_{(R_1,R_2),g,Q}$ and $N^{c=2}_{(R_1,R_2),g,Q}$ for various combinations of p_1 and p_2 are tabulated below.

$p_1 = 0 = p_2$

$$N^{c=1}_{\square,\square,0,\pm 1/2} = \pm 1$$

g	Q=-1	0
0	-1	1

$N^{c=1}_{\square\square,\square,g,Q}$ for hopf link

g	Q=0	1
0	1	-1

$N^{c=1}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},\square,g,Q}$ for hopf link

$p1 = 1 = p2$

g	Q=1/2	3/2
0	-1	1

$N^{c=1}_{\square,\square,g,Q}$ for hopf link

g	Q=3/2	5/2
0	-1	1

$N^{c=1}_{\square\square,\square,g,Q}$ for hopf link

g	Q=1/2	3/2	5/2
0	1	-5	4
1	0	-1	1

$N^{c=1}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},\square,g,Q}$ for hopf link

$p1 = 2 = p2$

g	Q=3/2	5/2
0	-1	1

$N^{c=1}_{\square,\square,g,Q}$ for hopf link

g	Q=2	3	4
0	-1	5	-4
1	0	1	-1

$N^{c=1}_{\square\square,\square,g,Q}$ for hopf link

g	Q=2	3	4
0	-4	13	-9
1	-1	7	-6
2	0	1	-1

$N^{c=1}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},\square,g,Q}$ for hopf link

$p1 = 2$, $p2 = 3$

g	Q=2	3
0	1	-1

g	Q=5/2	7/2	9/2
0	1	-5	4
1	0	-1	1

g	Q=5/2	7/2	9/2
0	4	-13	9
1	1	-7	6
2	0	-1	1

$N^{c=1}_{\square,\square,g,Q}$ for

hopf link

$N^{c=1}_{\square\square,\square,g,Q}$ for hopf link

$N^{c=1}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},\square,g,Q}$ for hopf link

7 Summary and Discussions

We have explicitly demonstrated the evaluation of framed knot and link invariants carrying composite representations in $U(N)$ Chern-Simons gauge theory. Particularly, we argued a specific choice for the $U(1)$ charge corresponding to the composite representations (2.10) so that the composite invariants for framed knots and links are polynomials in variables q, λ. Further, this direct method enabled us to verify generalised Rudolph's theorem for many framed knots (1.6).

The composite invariants was very essential to obtain the untwisted sector open topological string amplitude(3.4) on the orientifold of the resolved conifold geometry. Similar to Ooguri-Vafa conjecture (1.2), Marino [2] conjectured a form for the reformulated invariants (3.6) and the refined reformulated invariants (3.10). We have verified the conjecture for many framed knots and links and presented the reformulated invariants for few examples. The cross-cap $c = 0$ BPS integer coefficients (3.10) are also tabulated for these examples.

In earlier works [19–21], there was difficulty in seperating $c = 0$ and $c = 2$ contribution from the topological string free energy(1.4) but using the parity argument in variable $\sqrt{\lambda}$, the cross-cap $c = 1$ amplitude could be determined. With the present work on composite invariants following the approach [2], we can determine the unoriented topological string amplitude (1.4) by subtracting the untwisted sector contribution from the free energy of the open topological string theory on the orientifold. We have checked that the reformulated SO invariants obtained from the unoriented topological string free energy also obeys Marino's conjectured form (3.6). Further, the refined SO reformulated invariants obtained using eqn. (3.8) satisfies the conjectured form(3.11). We have tabulated the BPS integer invariants corresponding to cross-cap $c = 1$ and $c = 2$ obtained from reformulated invariants (3.11) for some framed knots and links. In particular, the $c = 1$ integer coefficients agrees with our earlier work [21]. Also, the BPS integer coefficients for the standard framing ($p = 0$) torus knots and torus links agrees with the results in Ref. [24]. The verification of Marino's conjectures for many framed knots and two-component framed links indirectly confirms that our choice of the $U(1)$ charge (2.10) for the composite representations is correct.

The Marino's conjectures, which we verified for some torus knots and torus links, should be obeyed by non-torus knots and non-torus links as well. The Chern-Simons approach requires the $SU(N)$ quantum Racah coefficients for the non-torus knot invariant evaluation. Unfortunately, these coefficients are not available in the literature. In Ref. [22], the $SU(N)$ quantum Racah coefficients for some representations could be determined using isotopy equivalence of knots enabling evaluation of non-torus knot invariants. We believe that there must be a similar approach of determining composite invariants for the non-torus knots.

It will be interesting to generalise these integrality properties in the context of Khovanov homology [28] and Kauffman homology [29]. We hope to report on this work in a future publication.

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Appendix

A $U(N)$ Composite Knot Invariants

A.1 For (R,S)=(2 horizonal box, single box)

Let us denote the composite representation $(\square\!\square\,,\square)$ by ρ_{02} whose highest weight is $\Lambda^{(N-1)}+2\Lambda^{(1)}$. The highest weights, quantum dimensions and the braiding eigenvalues corresponding to the irreducible representations $R_t \in \rho_{02} \otimes \rho_{02}$ are

R_t	highest weight	quantum dimension	λ_t
R_1	$2\Lambda^{(N-1)} + 4\Lambda^{(1)}$	$\frac{[N]^2[N-1][N+1][N+2][N+5]}{[4][3][2][2]}$	$q^{-3/2}\lambda^{3/2}$
R_2	$\Lambda^{(N)} + \Lambda^{(N-2)} + 4\Lambda^{(1)}$	$\frac{[N][N-1][N-2][N+1][N+3][N+4}{[4][3][2][2]}$	$-q^{-1/2}\lambda^{3/2}$
R_3	$2\Lambda^{(N-1)} + \Lambda^{(2)} + 2\Lambda^{(1)}$	$\frac{[N][N-1][N-2][N+1]^2[N+4]}{[4][2][2]}$	$-q^{1/2}\lambda^{3/2}$
R_4	$\Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(2)} + 2\Lambda^{(1)}$	$\frac{[N]^2[N-1][N+1][N+3]}{[4][2][2]}$	$q^{3/2}\lambda^{3/2}$
R_5	$\Lambda^{(N)} + \Lambda^{(N-1)} + 3\Lambda^{(1)}$	$\frac{[N][N-1][N+1][N+3]}{[3][2]}$	$q^{1/2}\lambda^2$
R_6	$\Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(2)} + \Lambda^{(1)}$	$\frac{[N]^2[N-2][N+2]}{[3]}$	$q^2\lambda^2$
R_7	$2\Lambda^{(N-1)} + 2\Lambda^{(2)}$	$\frac{[N][N-1]^2[N-2][N+2][N+3]}{[3][2][2][2]}$	$q^{3/2}\lambda^{3/2}$
R_8	$\Lambda^{(N)} + \Lambda^{(N-2)} + 2\Lambda^{(2)}$	$\frac{[N][N-2][N-3][N+1]^2[N+2]}{[3][2][2][2]}$	$-q^{5/2}\lambda^{3/2}$
R_9	$2\Lambda^{(N)} + 2\Lambda^{(1)}$	$\frac{[N][N+1]}{[2]}$	$q^{3/2}\lambda^{5/2}\lambda$
R_{10}	$2\Lambda^{(N)} + \Lambda^{(2)}$	$\frac{[N][N-1]}{[2]}$	$-q^{5/2}\lambda^{5/2}$
R_{11}	$\Lambda^{(N)} + \Lambda^{(N-1)} + 3\Lambda^{(1)}$	$\frac{[N][N-1][N+1][N+3]}{[3][2]}$	$-q^{1/2}\lambda^2$
R_{12}	$\Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(2)} + \Lambda^{(1)}$	$\frac{[N]^2[N-2][N+2]}{[3]}$	$-q^2\lambda^2$

Using the above table, we can evaluate directly the composite invariants for knots obtained as closure of two strand braids. We present few composite invariants for framed trefoil, framed

five-crossing knot 5_1 and framed seven-crossing knot 7_1 with arbitrary framing p:

$$\begin{aligned}
\mathcal{H}_{(\square\square,\square)}[\mathcal{K}_3] = &\ (dim_q\rho_{02})q^{-3}\lambda^{\frac{3p}{2}}\Big(\lambda^3 \\
&(q^8+2q^6+q^5+q^4+q^3+q^2+1) \\
&+\lambda^4(-1-q-2q^3-2q^4-q^5-2q^6-q^7-2q^9) \\
&+\lambda^5(q+2q^4+q^5-q^6+2q^7+q^{10})+\lambda^6(-q^5+q^6-2q^8+q^9) \\
&+\lambda^7(-q^7+q^8+q^9-q^{10})\Big)
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\mathcal{H}_{(\square\square,\square)}[\mathcal{K}_5] = &\ (dim_q\rho_{02})q^{-6}\lambda^{\frac{3p}{2}}\Big(\lambda^6(1+q^2+q^3+2q^4+q^5+2q^6+2q^7+3q^8+2q^9 \\
&+3q^{10}+2q^{11}+3q^{12}+q^{13}+2q^{14}+q^{16})+\lambda^7(-1-q-q^2-2q^3-2q^4-4q^5 \\
&-3q^6-4q^7-6q^8-5q^9-4q^{10}-6q^{11}-4q^{12}-3q^{13}-3q^{14}-3q^{15}-2q^{17}) \\
&+\lambda^8(q+q^3+q^4+2q^5+3q^6+2q^7+3q^8+6q^9+4q^{11}+2q^{13}+3q^{15}+q^{10} \\
&+4q^{12}+2q^{14}+q^{18})+\lambda^9(-q^6-q^7+q^8-2q^9-2q^{10}+2q^{11}-q^{13}+q^{14} \\
&-q^{16}+q^{17}-4q^{12}-q^{15})+\lambda^{10}(-q^9+2q^{10}-q^{11}+3q^{13}-2q^{14}-q^{15}-q^{12}+q^{16}) \\
&+\lambda^{11}(-q^{11}+2q^{12}-q^{13}-q^{14}-q^{16}+2q^{15})+\lambda^{12}(-q^{13}+q^{14}+q^{17}-q^{18})\Big)
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\mathcal{H}_{(\square\square,\square)}[\mathcal{K}_7] = &\ (dim_q\rho_{02})q^{-9}\lambda^{\frac{3p}{2}}\Big(\lambda^9(1+q^2+q^3+2q^4+q^5+3q^6 \\
&+2q^7+3q^8+3q^9+4q^{10}+3q^{11}+5q^{12} \\
&+4q^{13}+5q^{14}+4q^{15}+5q^{16}+3q^{17}+5q^{18}+2q^{19}+3q^{20}+q^{21}+2q^{22}+q^{24}) \\
&+\lambda^{10}(-1-q-q^2-2q^3-3q^4-4q^5-3q^6-6q^7-6q^8-7q^9-8q^{10}-9q^{11} \\
&-9q^{12}-11q^{13}-10q^{14}-10q^{15}-10q^{16}-10q^{17}-7q^{18}-8q^{19}-5q^{20}-5q^{21} \\
&-3q^{22}-3q^{23}-2q^{25})+\lambda^{11}(q+q^3+q^4+3q^5+2q^6+3q^7+5q^8+5q^9+5q^{10} \\
&+8q^{11}+6q^{12}+9q^{13}+8q^{14}+8q^{15}+7q^{16}+10q^{17}+5q^{18}+6q^{19}+6q^{20}+3q^{21} \\
&+2q^{22}+3q^{23}+q^{26})+\lambda^{12}(-q^6-q^8-2q^9-2q^{11}-3q^{12}-4q^{14}-2q^{15}-2q^{16} \\
&-3q^{17}-4q^{18}+q^{19}-3q^{20}-q^{21}+q^{22}-q^{23}-q^{24}+q^{25}) \\
&+\lambda^{13}(-q^{11}+2q^{12}-q^{13}-q^{14}+2q^{15}-q^{16}+q^{18}-q^{19}-q^{20}+2q^{21}-q^{22} \\
&-q^{23}+q^{24})+\lambda^{14}(-q^{13}+2q^{14}-q^{15}-q^{16}+2q^{17}-q^{18}+q^{20}-q^{21}-q^{22}+q^{23}) \\
&+\lambda^{15}(-q^{15}+2q^{16}-q^{17}-q^{18}+2q^{19}-q^{20})+\lambda^{16}(-q^{17}+2q^{18}-q^{19}-q^{20} \\
&+q^{21}+q^{23}-q^{24})+\lambda^{17}(-q^{19}+q^{20}+q^{25}-q^{26})\Big)
\end{aligned} \tag{A.3}$$

A.2 (R,S)=(two vertical box, single box)

Let us denote the composite representation $(\begin{smallmatrix}\square\\\square\end{smallmatrix},\square)$ by ρ_{03} and its highest weight is $\rho_{03}=\Lambda^{(N-1)}+\Lambda^{(2)}$. The representations R_t obtained from $\rho_{03}\otimes\rho_{03}$ and their quantum dimensions and the

signs of the braiding eigenvalues: ϵ_t are

$$
\begin{array}{ll}
R_1 = 2\Lambda^{(N-1)} + 2\Lambda^{(2)}; \epsilon_1 = 1 & R_2 = \Lambda^{(N)} + \Lambda^{(N-2)} + 2\Lambda^{(2)}; \epsilon_2 = -1 \\
R_3 = \Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(2)} + \Lambda^{(1)}; \epsilon_3 = 1 & R_4 = 2\Lambda^{(N)} + 2\Lambda^{(1)}; \epsilon_4 = 1 \\
R_5 = 2\Lambda^{(N-1)} + \Lambda^{(3)} + \Lambda^{(1)}; \epsilon_5 = -1 & R_6 = \Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(3)} + \Lambda^{(1)}; \epsilon_6 = 1 \\
R_7 = 2\Lambda^{(N)} + \Lambda^{(2)}; \epsilon_7 = -1 & R_8 = \Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(3)}; \epsilon_8 = 1 \\
R_9 = \Lambda^{(N-1)} + \Lambda^{(N-3)} + \Lambda^{(4)}; \epsilon_9 = 1 & R_{10} = \Lambda^{(N-1)} + \Lambda^{(N-3)} + \Lambda^{(4)}; \epsilon_{10} = -1 \\
R_{11} = \Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(2)} + \Lambda^{(1)}; \epsilon_{11} = -1 & R_{12} = \Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(3)}; \epsilon_{12} = -1
\end{array}
$$

For the above irreducible representations, quadratic casimir and quantum dimensions can be computed using eqns.(2.7,2.14). With this data, the polynomials of the framed knots carrying the composite representation can be computed. The composite invariants of some of the p-framed torus knots of type $(2, 2m+1)$ are:

$$
\begin{aligned}
\mathcal{H}_{(\begin{smallmatrix}\square\\\square\end{smallmatrix},\square)}[\mathcal{K}_3] = {}& (dim_q\rho_{03})q^{-7}\lambda^{\frac{3p}{2}} \Big(\lambda^3(q^2 + 2q^4 + q^5 + q^6 + q^7 + q^8 + q^{10}) \\
& +\lambda^4(-2q - q^3 - 2q^4 - q^5 - 2q^6 - 2q^7 - q^9 - q^{10}) \\
& +\lambda^5(1 + 2q^3 - q^4 + q^5 + 2q^6 + q^9) \\
& +\lambda^6(q - 2q^2 + q^4 - q^5) + \lambda^7(-1 + q + q^2 - q^3)\Big)
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
\mathcal{H}_{(\begin{smallmatrix}\square\\\square\end{smallmatrix},\square)}[\mathcal{K}_5] = {}& (dim_q\rho_{03})q^{-12}\lambda^{\frac{3p}{2}} \Big(\lambda^6(q^2 + 2q^4 + q^5 + 3q^6 + 2q^7 + 3q^8 + 2q^9 + 3q^{10} + 2q^{11} + 2q^{12} \\
& +q^{13} + 2q^{14} + q^{15} + q^{16} + q^{18}) + \lambda^7(-2q - 3q^3 - 3q^4 - 3q^5 - 4q^6 - 6q^7 - 4q^8 - 5q^9 \\
& -6q^{10} - 4q^{11} - 3q^{12} - 4q^{13} - 2q^{14} - 2q^{15} - q^{16} - q^{17} - q^{18}) \\
& +\lambda^8(1 + 3q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + q^8 + 6q^9 + 3q^{10} + 2q^{11} + 3q^{12} + 2q^{13} \\
& +q^{14} + q^{15} + q^{17}) + \lambda^9(q - q^2 - q^3 + q^4 - q^5 - 4q^6 + 2q^7 - 2q^8 - 2q^9 + q^{10} \\
& -q^{11} - q^{12}) + \lambda^{10}(q^2 - q^3 - 2q^4 - q^6 - q^7 + 2q^8 - q^9 + 3q^{10}) \\
& +\lambda^{11}(-q^2 + 2q^3 - q^4 - q^5 + 2q^6 - q^7) + \lambda^{12}(-1 + q + q^4 - q^5)\Big)
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
\mathcal{H}_{(\begin{smallmatrix}\square\\\square\end{smallmatrix},\square)}[\mathcal{K}_7] = {}& (dim_q\rho_{03})q^{-17}\lambda^{\frac{3p}{2}} \Big(\lambda^9(q^2 + 2q^4 + q^5 + 3q^6 + 2q^7 + 5q^8 + 3q^9 + 5q^{10} + 4q^{11} + 5q^{12} \\
& +4q^{13} + 5q^{14} + 3q^{15} + 4q^{16} + 3q^{17} + 3q^{18} + 2q^{19} + 3q^{20} + q^{21} + 2q^{22} + q^{23} + q^{24} + q^{26}) \\
& +\lambda^{10}(-3q^3 - 3q^4 - 5q^5 - 5q^6 - 2q - 8q^7 - 7q^8 - 10q^9 - 10q^{10} - 10q^{11} - 10q^{12} \\
& -11q^{13} - 9q^{14} - 9q^{15} - 8q^{16} - 7q^{17} - 6q^{18} - 6q^{19} - 3q^{20} - 4q^{21} - 3q^{22} - 2q^{23} \\
& -q^{24} - q^{25} - q^{26}) + \lambda^{11}(1 + 3q^3 + 2q^4 + 3q^5 + 6q^6 + 6q^7 + 5q^8 + 10q^9 + 7q^{10} \\
& +8q^{11} + 8q^{12} + 9q^{13} + 6q^{14} + 8q^{15} + 5q^{16} + 5q^{17} + 5q^{18} + 3q^{19} + 2q^{20} + 3q^{21} \\
& +q^{22} + q^{23} + q^{25}) + \lambda^{12}(q - q^2 - q^3 + q^4 - q^5 - 3q^6 + q^7 - 4q^8 - 3q^9 - 2q^{10} \\
& -2q^{11} - 4q^{12} - 3q^{14} - 2q^{15} - 2q^{17} - q^{18} - q^{20}) + \lambda^{13}(q^2 - q^3 - q^4 + 2q^5 - q^6 \\
& -q^7 + q^8 - q^{10} + 2q^{11} - q^{12} - q^{13} + 2q^{14} - q^{15}) + \lambda^{14}(q^3 - q^4 - q^5 + q^6 - q^8
\end{aligned}
$$

$$
\begin{aligned}
&+2q^9-q^{10}-q^{11}+2q^{12}-q^{13})-\lambda^{15}(q^6-2q^7+q^8+q^9-2q^{10}+q^{11})\\
&-\lambda^{16}(+q^2-q^3-q^5+q^6+q^7-2q^8+q^9)+\lambda^{17}(-1+q+q^6-q^7)\Big)
\end{aligned}
\qquad \text{(A.6)}
$$

A.3 (R,S)=(two horizontal box, two horizontal box)

Let us denote the composite representation $(\square\!\square\,,\square\!\square\,)$ by ρ_{04} whose highest weight is

$$
\rho_{04}=2\Lambda^{(N-1)}+2\Lambda^{(1)} \qquad \text{(A.7)}
$$

The highest weights of the ireducible representations R_t obtained from $\rho_{04}\otimes\rho_{04}$ and the signs of the braiding eigenvalues ϵ_t are

R_t	Highest weight	ϵ_t	R_t	Highest weight	ϵ_t
R_1	$4\Lambda^{(N-1)}+4\Lambda^{(1)}$	1	R_2	$\Lambda^{(N)}+2\Lambda^{(N-1)}+\Lambda^{(N-2)}+4\Lambda^{(1)}$	-1
R_3	$2\Lambda^{(N)}+2\Lambda^{(N-2)}+4\Lambda^{(1)}$	1	R_4	$\Lambda^{(N)}+3\Lambda^{(N-1)}+3\Lambda^{(1)}$	1
R_5	$2\Lambda^{(N)}+\Lambda^{(N-1)}+\Lambda^{(N-2)}+3\Lambda^{(1)}$	1	R_6	$\Lambda^{(N)}+2\Lambda^{(N-1)}+\Lambda^{(N-2)}+\Lambda^{(2)}+2\Lambda^{(1)}$	1
R_7	$2\Lambda^{(N)}+2\Lambda^{(N-2)}+\Lambda^{(2)}+2\Lambda^{(1)}$	-1	R_8	$2\Lambda^{(N)}+2\Lambda^{(N-1)}+2\Lambda^{(1)}$	1
R_9	$2\Lambda^{(N)}+\Lambda^{(N-1)}+\Lambda^{(N-2)}+\Lambda^{(2)}+\Lambda^{(1)}$	-1	R_{10}	$2\Lambda^{(N)}+2\Lambda^{(N-2)}+2\Lambda^{(2)}$	1
R_{11}	$2\Lambda^{(N)}+2\Lambda^{(N-1)}+\Lambda^{(2)}$	-1	R_{12}	$3\Lambda^{(N)}+\Lambda^{(N-2)}+\Lambda^{(2)}$	1
R_{13}	$3\Lambda^{(N)}+\Lambda^{(N-1)}+\Lambda^{(1)}$	1	R_{14}	$4\Lambda^{(N)}$	1
R_{15}	$4\Lambda^{(N-1)}+\Lambda^{(2)}+2\Lambda^{(1)}$	-1	R_{16}	$4\Lambda^{(N-1)}+2\Lambda^{(2)}$	1
R_{17}	$\Lambda^{(N)}+3\Lambda^{(N-1)}+3\Lambda^{(1)}$	-1	R_{18}	$2\Lambda^{(N)}+\Lambda^{(N-1)}+\Lambda^{(N-2)}+3\Lambda^{(1)}$	-1
R_{19}	$\Lambda^{(N)}+3\Lambda^{(N-1)}+\Lambda^{(2)}+\Lambda^{(1)}$	1	R_{20}	$\Lambda^{(N)}+3\Lambda^{(N-1)}+\Lambda^{(2)}+\Lambda^{(1)}$	-1
R_{21}	$\Lambda^{(N)}+2\Lambda^{(N-1)}+\Lambda^{(N-2)}+2\Lambda^{(2)}$	-1	R_{22}	$2\Lambda^{(N)}+2\Lambda^{(N-1)}+2\Lambda^{(1)}$	1
R_{23}	$2\Lambda^{(N)}+2\Lambda^{(N-1)}+2\Lambda^{(1)}$	-1	R_{24}	$2\Lambda^{(N)}+\Lambda^{(N-1)}+\Lambda^{(N-2)}+\Lambda^{(2)}+\Lambda^{(1)}$	1
R_{25}	$3\Lambda^{(N)}+\Lambda^{(N-2)}+2\Lambda^{(1)}$	-1	R_{26}	$3\Lambda^{(N)}+\Lambda^{(N-1)}+\Lambda^{(1)}$	-1

Using the above data, the composite invariants can be computed. The composite polynomials for some of the framed p knots are

$$
\begin{aligned}
\mathcal{H}_{(\square\!\square\,,\square\!\square\,)}[\mathcal{K}_3]=\;&(dim_q\rho_{04})q^{-4}\lambda^{2p}\Big(\lambda^4(1+2q^3+2q^4+3q^6+2q^7+q^8+2q^9+2q^{10}+q^{12})\\
&+\lambda^5(-2q-2q^2+q^3-4q^4-6q^5-3q^7-6q^8-q^9-2q^{10}-4q^{11}-q^{13}-2q^{14})\\
&+\lambda^6(q^2+4q^3-q^4+7q^6+2q^7-3q^8+6q^9+4q^{10}-2q^{11}+3q^{12}+2q^{13}\\
&-2q^{14}+2q^{15}+q^{16})+\lambda^7(-2q^4-q^5+4q^6-3q^7-6q^8+6q^9-8q^{11}+4q^{12}\\
&+2q^{13}-6q^{14}+q^{15}+2q^{16}-q^{27})+\lambda^8(q^6-2q^7-q^8+6q^9-3q^{10}-6q^{11}\\
&+9q^{12}-8q^{14}+4q^{15}+4q^{16}-2q^{17}-q^{18})+\lambda^9(q^9-2q^{10}-q^{11}+6q^{12}-3q^{13}\\
&-6q^{14}+6q^{15}+2q^{16}-4q^{17}+q^{19})\\
&+\lambda^{10}(+q^{12}-2q^{13}-q^{14}+4q^{15}-q^{16}-2q^{17}+q^{18})\Big)
\end{aligned}
\qquad \text{(A.8)}
$$

$$
\begin{aligned}
\mathcal{H}_{(\square\square,\square\square)}[\mathcal{K}_5] = {} & (dim_q\rho_{04})q^{-8}\lambda^{2p}\Big(\lambda^8(1+2q^3+2q^4+3q^6+4q^7+3q^8+4q^9+6q^{10}+4q^{11}+7q^{12}\\
& +6q^{13}+5q^{14}+6q^{15}+7q^{16}+4q^{17}+5q^{18}+4q^{19}+3q^{20}+2q^{21}+2q^{22}+q^{24})\\
& +\lambda^9(-2q-2q^2+q^3-4q^4-8q^5-2q^6-5q^7-14q^8-9q^9-10q^{10}-17q^{11}\\
& -12q^{12}-15q^{13}-20q^{14}-12q^{15}-14q^{16}-19q^{17}-10q^{18}-9q^{19}-12q^{20}-6q^{21}\\
& -4q^{22}-6q^{23}-2q^{24}-q^{25}-2q^{26})+\lambda^{10}(q^2+4q^3-q^4+11q^6+6q^7-q^8\\
& +16q^9+13q^{10}+6q^{11}+18q^{12}+16q^{13}+10q^{14}+24q^{15}+15q^{16}+8q^{17}+21q^{18}\\
& +14q^{19}+2q^{20}+14q^{21}+8q^{22}+6q^{24}+4q^{25}-2q^{26}+2q^{27}+q^{28})\\
& +\lambda^{11}(-2q^4-q^5+4q^6-5q^7-8q^8+6q^9-4q^{10}-14q^{11}+2q^{12}-5q^{13}-16q^{14}\\
& -q^{15}-8q^{16}-16q^{17}+2q^{18}-6q^{19}-16q^{20}+4q^{21}-2q^{22}-12q^{23}+4q^{24}+q^{25}\\
& -6q^{26}+2q^{27}+2q^{28}-q^{29})+\lambda^{12}(q^6-2q^7-q^8+6q^9-3q^{10}-6q^{11}+11q^{12}\\
& -9q^{14}+12q^{15}+3q^{16}-10q^{17}+12q^{18}+2q^{19}-9q^{20}+12q^{21}+2q^{22}-12q^{23}\\
& +10q^{24}+2q^{25}-10q^{26}+4q^{27}+3q^{28}-2q^{29})+\lambda^{13}(q^9-2q^{10}-q^{11}+6q^{12}\\
& -4q^{13}-6q^{14}+11q^{15}-2q^{16}-11q^{17}+12q^{18}-12q^{20}+11q^{21}-11q^{23}\\
& +12q^{24}+q^{25}-12q^{26}+7q^{27}+4q^{28}-4q^{29})+\lambda^{14}(+q^{12}-2q^{13}-q^{14}+6q^{15}\\
& -4q^{16}-6q^{17}+11q^{18}-2q^{19}-11q^{20}+12q^{21}-12q^{23}+11q^{24}-11q^{26}+8q^{27}\\
& +4q^{28}-4q^{29}+q^{30}-q^{32})+\lambda^{15}(q^{15}-2q^{16}-q^{17}+6q^{18}-4q^{19}-6q^{20}+11q^{21}\\
& -2q^{22}-10q^{23}+12q^{24}-12q^{26}+6q^{27}+4q^{28}-3q^{29}-q^{31}+q^{33})\\
& +\lambda^{16}(q^{18}-2q^{19}-q^{20}+6q^{21}-4q^{22}-6q^{23}+11q^{24}-2q^{25}-11q^{26}+8q^{27}\\
& +5q^{28}-4q^{29}-2q^{30}+q^{32})+\lambda^{17}(q^{21}-2q^{22}-q^{23}+6q^{24}-4q^{25}-6q^{26}+8q^{27}\\
& +2q^{28}-5q^{29}+q^{31})+\lambda^{18}(q^{24}-2q^{25}-q^{26}+4q^{27}-q^{28}-2q^{29}+q^{30})\Big)
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
\mathcal{H}_{(\square\square,\square\square)}[\mathcal{K}_7] = {} & (dim_q\rho_{04})q^{-12}\lambda^{2p}\Big(\lambda^{12}(1+2q^3+2q^4+3q^6+4q^7+3q^8+4q^9+6q^{10}+6q^{11}+9q^{12}\\
& +8q^{13}+9q^{14}+12q^{15}+13q^{16}+10q^{17}+15q^{18}+14q^{19}+13q^{20}+14q^{21}+15q^{22}\\
& +12q^{23}+14q^{24}+12q^{25}+11q^{26}+10q^{27}+9q^{28}+6q^{29}+7q^{30}+4q^{31}+3q^{32}\\
& +2q^{33}+2q^{34}+q^{36})+\lambda^{13}(-2q-2q^2+q^3-4q^4-8q^5-2q^6-5q^7-14q^8\\
& -11q^9-12q^{10}-19q^{11}-20q^{12}-25q^{13}-30q^{14}-26q^{15}-34q^{16}-43q^{17}\\
& -34q^{18}-38q^{19}-48q^{20}-40q^{21}-40q^{22}-46q^{23}-38q^{24}-36q^{25}-38q^{26}\\
& -30q^{27}-26q^{28}-26q^{29}-18q^{30}-15q^{31}-14q^{32}-8q^{33}-6q^{34}-6q^{35}-2q^{36}\\
& -q^{37}-2q^{38})+\lambda^{14}(q^2+4q^3-q^4+11q^6+6q^7-q^8+16q^9+17q^{10}+10q^{11}\\
& +22q^{12}+26q^{13}+22q^{14}+40q^{15}+33q^{16}+30q^{17}+54q^{18}+48q^{19}+33q^{20}\\
& +60q^{21}+54q^{22}+38q^{23}+54q^{24}+50q^{25}+32q^{26}+46q^{27}+36q^{28}+22q^{29}\\
& +29q^{30}+24q^{31}+9q^{32}+16q^{33}+10q^{34}+2q^{35}+6q^{36}+4q^{37}-2q^{38}+2q^{39}\\
& +q^{40})+\lambda^{15}(-2q^4-q^5+4q^6-5q^7-8q^8+6q^9-4q^{10}-16q^{11}-7q^{13}-20q^{14}
\end{aligned}
$$

$$
\begin{aligned}
&-5q^{15}-18q^{16}-28q^{17}-6q^{18}-26q^{19}-40q^{20}-9q^{21}-26q^{22}-45q^{23}-12q^{24}\\
&-22q^{25}-40q^{26}-10q^{27}-18q^{28}-30q^{29}-4q^{30}-10q^{31}-20q^{32}+q^{33}-2q^{34}\\
&-11q^{35}+4q^{36}+q^{37}-6q^{38}+2q^{39}+2q^{40}-q^{41})+\lambda^{16}(q^{6}-2q^{7}-q^{8}+6q^{9}\\
&-3q^{10}-6q^{11}+11q^{12}-9q^{14}+12q^{15}+3q^{16}-8q^{17}+16q^{18}+2q^{19}-7q^{20}\\
&+22q^{21}+3q^{22}-12q^{23}+26q^{24}+6q^{25}-14q^{26}+22q^{27}+9q^{28}-14q^{29}\\
&+17q^{30}+4q^{31}-12q^{32}+12q^{33}+2q^{34}-12q^{35}+9q^{36}+2q^{37}-9q^{38}+4q^{39}\\
&+3q^{40}-2q^{41})+\lambda^{17}(q^{9}-2q^{10}-q^{11}+6q^{12}-4q^{13}-6q^{14}+11q^{15}-2q^{16}\\
&-11q^{17}+12q^{18}-12q^{20}+13q^{21}-2q^{22}-13q^{23}+18q^{24}-4q^{25}-18q^{26}\\
&+21q^{27}-21q^{29}+18q^{30}+4q^{31}-18q^{32}+14q^{33}+2q^{34}-14q^{35}+12q^{36}\\
&+q^{37}-12q^{38}+6q^{39}+4q^{40}-3q^{41})+\lambda^{18}(+q^{12}-2q^{13}-q^{14}+6q^{15}-4q^{16}\\
&-6q^{17}+11q^{18}-2q^{19}-11q^{20}+12q^{21}-12q^{23}+13q^{24}-2q^{25}-13q^{26}\\
&+18q^{27}-4q^{28}-18q^{29}+21q^{30}-21q^{32}+18q^{33}+4q^{34}-18q^{35}+14q^{36}\\
&+2q^{37}-14q^{38}+8q^{39}+4q^{40}-4q^{41})+\lambda^{19}(q^{15}-2q^{16}-q^{17}+6q^{18}-4q^{19}\\
&-6q^{20}+11q^{21}-2q^{22}-11q^{23}+12q^{24}-12q^{26}+13q^{27}-2q^{28}-13q^{29}\\
&+18q^{30}-4q^{31}-18q^{32}+21q^{33}-21q^{35}+18q^{36}+4q^{37}-18q^{38}+9q^{39}\\
&+6q^{40}-4q^{41}-q^{43})+\lambda^{20}(q^{18}-2q^{19}-q^{20}+6q^{21}-4q^{22}-6q^{23}+11q^{24}\\
&-2q^{25}-11q^{26}+12q^{27}-12q^{29}+13q^{30}-2q^{31}-13q^{32}+18q^{33}-4q^{34}\\
&-18q^{35}+21q^{36}-21q^{38}+12q^{39}+9q^{40}-6q^{41}-q^{42}+q^{44}-q^{46})\\
&+\lambda^{21}(q^{21}-2q^{22}-q^{23}+6q^{24}-4q^{25}-6q^{26}+11q^{27}-2q^{28}-11q^{29}+12q^{30}\\
&-12q^{32}+14q^{33}-2q^{34}-13q^{35}+18q^{36}-4q^{37}-18q^{38}+14q^{39}+6q^{40}-9q^{41}\\
&+2q^{43}-q^{45}+q^{47})+\lambda^{22}(q^{24}-2q^{25}-q^{26}+6q^{27}-4q^{28}-6q^{29}+11q^{30}\\
&-2q^{31}-11q^{32}+12q^{33}-12q^{35}+13q^{36}-2q^{37}-13q^{38}+12q^{39}+3q^{40}-6q^{41}\\
&+2q^{42}-2q^{44}+q^{46})+\lambda^{23}(q^{27}-2q^{28}-q^{29}+6q^{30}-4q^{31}-6q^{32}+11q^{33}\\
&-2q^{34}-11q^{35}+12q^{36}-12q^{38}+8q^{39}+4q^{40}-3q^{41}-2q^{43}+q^{45})\\
&+\lambda^{24}(q^{30}-2q^{31}-q^{32}+6q^{33}-4q^{34}-6q^{35}+11q^{36}-2q^{37}-11q^{38}+8q^{39}\\
&+5q^{40}-4q^{41}-2q^{42}+q^{44})+\lambda^{25}(q^{33}-2q^{34}-q^{35}+6q^{36}-4q^{37}-6q^{38}\\
&+8q^{39}+2q^{40}-5q^{41}+q^{43})+\lambda^{26}(q^{36}-2q^{37}-q^{38}+4q^{39}-q^{40}-2q^{41}+q^{42})\Big)
\end{aligned}
\tag{A.10}
$$

A.4 (R,S)=(two vertical box, two vertical box)

Let us denote the composite representation $(\boxminus,\boxminus)$ by ρ_{05} and its highest weight is $\rho_{05}=\Lambda^{(N-2)}+\Lambda^{(2)}$. The representations R_t obtained from $\rho_{05}\otimes\rho_{05}$ and the signs of the braiding eigenvalues

ϵ_t are

$$
\begin{array}{ll}
R_1 = \Lambda^{(N-2)} + \Lambda^{(N-2)} + \Lambda^{(2)} + \Lambda^{(2)}; \epsilon_1 = 1 & R_2 = \Lambda^{(N-1)} + \Lambda^{(N-3)} + \Lambda^{(2)} + \Lambda^{(2)}; \epsilon_2 = -1 \\
R_3 = \Lambda^{(N)} + \Lambda^{(N-4)} + \Lambda^{(2)} + \Lambda^{(2)}; \epsilon_3 = 1 & R_4 = \Lambda^{(N-1)} + \Lambda^{(N-2)} + \Lambda^{(2)} + \Lambda^{(1)}; \epsilon_4 = 1 \\
R_5 = \Lambda^{(N)} + \Lambda^{(N-3)} + \Lambda^{(2)} + \Lambda^{(1)}; \epsilon_5 = 1 & R_6 = \Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(1)} + \Lambda^{(1)}; \epsilon_6 = -1 \\
R_7 = \Lambda^{(N-2)} + \Lambda^{(N-2)} + \Lambda^{(3)} + \Lambda^{(1)}; \epsilon_7 = -1 & R_8 = \Lambda^{(N-1)} + \Lambda^{(N-3)} + \Lambda^{(3)} + \Lambda^{(1)}; \epsilon_8 = 1 \\
R_9 = \Lambda^{(N)} + \Lambda^{(N-4)} + \Lambda^{(3)} + \Lambda^{(1)}; \epsilon_9 = -1 & R_{10} = \Lambda^{(N-1)} + \Lambda^{(N-1)} + \Lambda^{(2)}; \epsilon_{10} = -1 \\
R_{11} = \Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(2)}; \epsilon_{11} = 1 & R_{12} = \Lambda^{(N-1)} + \Lambda^{(N-1)} + \Lambda^{(1)} + \Lambda^{(1)}; \epsilon_{12} = 1 \\
R_{13} = \Lambda^{(N-1)} + \Lambda^{(N-2)} + \Lambda^{(3)}; \epsilon_{13} = 1 & R_{14} = \Lambda^{(N)} + \Lambda^{(N-3)} + \Lambda^{(3)}; \epsilon_{14} = 1 \\
R_{15} = \Lambda^{(N-2)} + \Lambda^{(N-2)} + \Lambda^{(4)}; \epsilon_{15} = 1 & R_{16} = \Lambda^{(N-1)} + \Lambda^{(N-3)} + \Lambda^{(4)}; \epsilon_{16} = -1 \\
R_{17} = \Lambda^{(N)} + \Lambda^{(N-4)} + \Lambda^{(4)}; \epsilon_{17} = 1 & R_{18} = \Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(1)}; \epsilon_{18} = 1 \\
R_{19} = 2\Lambda^{(N)}; \epsilon_{19} = 1 & R_{20} = \Lambda^{(N-1)} + \Lambda^{(N-2)} + \Lambda^{(2)} + \Lambda^{(1)}; \epsilon_{20} = -1 \\
R_{21} = \Lambda^{(N)} + \Lambda^{(N-3)} + \Lambda^{(2)} + \Lambda^{(1)}; \epsilon_{21} = -1 & R_{22} = \Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(2)}; \epsilon_{22} = -1 \\
R_{23} = \Lambda^{(N)} + \Lambda^{(N-2)} + \Lambda^{(2)}; \epsilon_{23} = 1 & R_{24} = \Lambda^{(N-1)} + \Lambda^{(N-2)} + \Lambda^{(3)}; \epsilon_{24} = -1 \\
R_{25} = \Lambda^{(N)} + \Lambda^{(N-3)} + \Lambda^{(3)}; \epsilon_{25} = -1 & R_{26} = \Lambda^{(N)} + \Lambda^{(N-1)} + \Lambda^{(1)}; \epsilon_{26} = -1
\end{array}
$$

The composite invariants of framed knots and links can be computed using the above data. Some of the framed p knot polynomials are

$$
\begin{aligned}
\mathcal{H}_{(\square,\square)}[\mathcal{K}_3] = {} & (dim_q\rho_{05})q^{-15}\lambda^{2p}\Big(\lambda^4(q^7+2q^9+2q^{10}+q^{11}+2q^{12}+3q^{13}+2q^{15}+2q^{16}+q^{19}) \\
& +\lambda^5(-2q^5-q^6-4q^8-2q^9-q^{10}-6q^{11}-3q^{12}-6q^{14}-4q^{15}+q^{16}-2q^{17}-2q^{18}) \\
& +\lambda^6(q^3+2q^4-2q^5+2q^6+3q^7-2q^8+4q^9+6q^{10}-3q^{11}+2q^{12}+7q^{13}-q^{15} \\
& +4q^{16}+q^{17})+\lambda^7(-q^2+2q^3+q^4-6q^5+2q^6+4q^7-8q^8+6q^{10}-6q^{11}-3q^{12} \\
& +4q^{13}-q^{14}-2q^{15})+\lambda^8(-q-2q^2+4q^3+4q^4-8q^5+9q^7-6q^8-3q^9+6q^{10} \\
& -q^{11}-2q^{12}+q^{13})+\lambda^9(1-4q^2+2q^3+6q^4-6q^5-3q^6+6q^7-q^8-2q^9+q^{10}) \\
& +\lambda^{10}(q-2q^2-q^3+4q^4-q^5-2q^6+q^7)\Big) \qquad \text{(A.11)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_{(\square,\square)}[\mathcal{K}_5] = {} & (dim_q\rho_{05})q^{-25}\lambda^{2p}\Big(\lambda^8(q^{-16}+2q^{-14}+2q^{-13}+3q^{-12}+4q^{-11}+5q^{-10}+4q^{-9}+7q^{-8} \\
& +6q^{-7}+5q^{-6}+6q^{-5}+7q^{-4}+4q^{-3}+6q^{-2}+4q^{-1}+3+4q+3q^2+2q^4+2q^5+q^8) \\
& +\lambda^9(-2q^{-18}-q^{-17}-2q^{-16}-6q^{-15}-4q^{-14}-6q^{-13}-12q^{-12}-9q^{-11}-10q^{-10} \\
& -19q^{-9}-14q^{-8}-12q^{-7}-20q^{-6}-15q^{-5}-12q^{-4}-17q^{-3}-10q^{-2}-9q^{-1} \\
& -14-5q-2q^2-8q^3-4q^4+q^5-2q^6-2q^7)+\lambda^{10}(q^{-20}+2q^{-19}-2q^{-18} \\
& +4q^{-17}+6q^{-16}+8q^{-14}+14q^{-13}+2q^{-12}+14q^{-11}+21q^{-10}+8q^{-9}+15q^{-8} \\
& +24q^{-7}+10q^{-6}+16q^{-5}+18q^{-4}+6q^{-3}+13q^{-2}+16q^{-1}-1+6q+11q^2 \\
& -q^4+4q^5+q^6)+\lambda^{11}(-q^{-21}+2q^{-20}+2q^{-19}-6q^{-18}+q^{-17}+4q^{-16}-12q^{-15} \\
& -2q^{-14}+4q^{-13}-16q^{-12}-6q^{-11}+2q^{-10}-16q^{-9}-8q^{-8}-q^{-7}-16q^{-6}-5q^{-5} \\
& +2q^{-4}-14q^{-3}-4q^{-2}+6q^{-1}-8-5q+4q^2-q^3-2q^4)-\lambda^{12}(2q^{-21}+3q^{-20}
\end{aligned}
$$

$$+4q^{-19}-10q^{-18}+2q^{-17}+10q^{-16}-12q^{-15}+2q^{-14}+12q^{-13}-9q^{-12}+2q^{-11}$$
$$+12q^{-10}-10q^{-9}+3q^{-8}+12q^{-7}-9q^{-6}+11q^{-4}-6q^{-3}-3q^{-2}+6q^{-1}-1$$
$$-2q+q^{2})+\lambda^{13}(-4q^{-21}+4q^{-20}+7q^{-19}-12q^{-18}+q^{-17}+12q^{-16}-11q^{-15}$$
$$+11q^{-13}-12q^{-12}+12q^{-10}-11q^{-9}-2q^{-8}+11q^{-7}-6q^{-6}-4q^{-5}+6q^{-4}$$
$$-q^{-3}-2q^{-2}+q^{-1})+\lambda^{14}(-q^{-24}+q^{-22}-4q^{-21}+4q^{-20}+8q^{-19}-11q^{-18}$$
$$+11q^{-16}-12q^{-15}+12q^{-13}-11q^{-12}-2q^{-11}+11q^{-10}-6q^{-9}-4q^{-8}+6q^{-7}$$
$$-q^{-6}-2q^{-5}+q^{-4})+\lambda^{15}(q^{-25}-q^{-23}-3q^{-21}+4q^{-20}+6q^{-19}-12q^{-18}$$
$$+12q^{-16}-10q^{-15}-2q^{-14}+11q^{-13}-6q^{-12}-4q^{-11}+6q^{-10}-q^{-9}-2q^{-8}+q^{-7})$$
$$+\lambda^{16}(q^{-24}-2q^{-22}-4q^{-21}+5q^{-20}+8q^{-19}-11q^{-18}-2q^{-17}+11q^{-16}-6q^{-15}$$
$$-4q^{-14}+6q^{-13}-q^{-12}-2q^{-11}+q^{-10})+\lambda^{17}(q^{-23}-5q^{-21}+2q^{-20}+8q^{-19}$$
$$-6q^{-18}-4q^{-17}+6q^{-16}-q^{-15}-2q^{-14}+q^{-13})+\lambda^{18}(q^{-22}-2q^{-21}-q^{-20}$$
$$+4q^{-19}-q^{-18}-2q^{-17}+q^{-16})\Big) \tag{A.12}$$

These composite invariants $\mathcal{H}_{(R,R)}[\mathcal{K}]$ and the corresponding $SO(N)$ invariants $\mathcal{G}_R[\mathcal{K}]$ given in appendix A of Ref. [21] satisfy the generalised Rudolph theorem (1.6).

B $SO(N)$ Reformulated Invariants $g_{R_1,R_2,\ldots,R_r}(q,\lambda)$

Here we present the SO reformulated invariants for framed knots and links.

B.1 p Framed Torus Knot $(2,5)$

$$g_{\square} = (-1)^p\, q^{-1}\, \lambda^{\frac{p}{2}+1} \left(1-q^2\,(-1+\lambda)-\lambda+q\left(1-\lambda+\lambda^2\right)\right) \tag{B.1}$$

$$\begin{aligned} g_{\square\square} \;=\; & \frac{-\lambda^p}{2\,(-1+q)^2\,q^4}\Big((-1+\lambda)\,\lambda\,\Big(2\,q^{\frac{5}{2}+p}\sqrt{\lambda}+2\,q^{\frac{9}{2}+p}\sqrt{\lambda}-2\,q^{\frac{19}{2}+p}\sqrt{\lambda} \\ & +2\,q^{\frac{5}{2}}\,(-1+\lambda)\sqrt{\lambda}-2\,q^{\frac{15}{2}}\,(-1+\lambda)\sqrt{\lambda}+2\,q^{\frac{7}{2}}\,(1-2\,\lambda)\,\lambda^{\frac{3}{2}} \\ & +q\,(-1+\lambda)\,\lambda^2+q^9\,(-1+\lambda)\,\lambda^2+2\,q^{\frac{11}{2}+p}\,\lambda^{\frac{5}{2}}-2\,q^{\frac{25}{2}+p}\,\lambda^{\frac{5}{2}}+2\,q^{9+p}\,\lambda^3 \\ & +2\,q^{13+p}\,\lambda^3-(-1+\lambda)\,\lambda^3-q^{10}\,(-1+\lambda)\,\lambda^3-2\,q^{\frac{7}{2}+p}\sqrt{\lambda}\,(1+\lambda) \\ & -2\,q^{\frac{13}{2}+p}\,\lambda^{\frac{3}{2}}\,(1+\lambda)+2\,q^{\frac{15}{2}+p}\,\lambda^{\frac{3}{2}}\,(1+\lambda)-2\,q^{\frac{21}{2}+p}\,\lambda^{\frac{5}{2}}\,(1+\lambda) \\ & -2\,q^{10+p}\,\lambda^2\,(1+\lambda)^2-2\,q^{12+p}\,\lambda^2\,(1+\lambda)^2+2\,q^{\frac{13}{2}}\,\lambda^{\frac{3}{2}}\,(-1+2\,\lambda) \\ & +2\,q^{\frac{23}{2}+p}\,\lambda^{\frac{3}{2}}\left(1+\lambda+\lambda^2\right)+4\,q^{11+p}\,\lambda^2\left(1+\lambda+\lambda^2\right) \\ & +q^2\,\lambda\left(-1+\lambda+2\,\lambda^2-2\,\lambda^3\right)+q^8\,\lambda\left(-1+\lambda+2\,\lambda^2-2\,\lambda^3\right) \\ & +2\,q^{\frac{9}{2}}\sqrt{\lambda}\left(-1+\lambda+\lambda^2+\lambda^3\right)-2\,q^{\frac{11}{2}}\sqrt{\lambda}\left(-1+\lambda+\lambda^2+\lambda^3\right) \end{aligned}$$

$$
\begin{aligned}
&+q^{3}\,\left(1-\lambda-2\,\lambda^{2}-\lambda^{4}+\lambda^{5}\right)+q^{7}\,\left(1-\lambda-2\,\lambda^{2}-\lambda^{4}+\lambda^{5}\right)\\
&+q^{4}\,\lambda\,\left(-2+2\,\lambda+6\,\lambda^{2}-2\,\lambda^{3}+\lambda^{4}+\lambda^{5}+\lambda^{6}+\lambda^{7}\right)\\
&+q^{6}\,\lambda\,\left(-2+2\,\lambda+6\,\lambda^{2}-2\,\lambda^{3}+\lambda^{4}+\lambda^{5}+\lambda^{6}+\lambda^{7}\right)\\
&-2\,q^{5}\,\left(-1+2\,\lambda^{2}+2\,\lambda^{4}+\lambda^{6}+\lambda^{7}+\lambda^{8}\right)\Big)\Big)
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
g_{\substack{\square\\ \square}} = {}& \frac{-\lambda^{p}}{2\,(-1+q)^{2}}\,\Big(q^{-7-p}\,(-1+\lambda)\,\lambda\,\Big(2\,q^{\frac{7}{2}}\,\sqrt{\lambda}-2\,q^{\frac{17}{2}}\,\sqrt{\lambda}-2\,q^{\frac{21}{2}}\,\sqrt{\lambda}\\
&+2\,q^{\frac{11}{2}+p}\,(-1+\lambda)\,\sqrt{\lambda}-2\,q^{\frac{21}{2}+p}\,(-1+\lambda)\,\sqrt{\lambda}+q^{4+p}\,(-1+\lambda)\,\lambda^{2}\\
&+q^{12+p}\,(-1+\lambda)\,\lambda^{2}+2\,\sqrt{q}\,\lambda^{\frac{5}{2}}-2\,q^{\frac{15}{2}}\,\lambda^{\frac{5}{2}}+2\,\lambda^{3}+2\,q^{4}\,\lambda^{3}\\
&-q^{3+p}\,(-1+\lambda)\,\lambda^{3}-q^{13+p}\,(-1+\lambda)\,\lambda^{3}+2\,q^{\frac{19}{2}}\,\sqrt{\lambda}\,(1+\lambda)\\
&-2\,q^{\frac{11}{2}}\,\lambda^{\frac{3}{2}}\,(1+\lambda)+2\,q^{\frac{13}{2}}\,\lambda^{\frac{3}{2}}\,(1+\lambda)+2\,q^{\frac{5}{2}}\,\lambda^{\frac{5}{2}}\,(1+\lambda)-2\,q\,\lambda^{2}\,(1+\lambda)^{2}\\
&-2\,q^{3}\,\lambda^{2}\,(1+\lambda)^{2}-2\,q^{\frac{13}{2}+p}\,\lambda^{\frac{3}{2}}\,(-1+2\,\lambda)+2\,q^{\frac{19}{2}+p}\,\lambda^{\frac{3}{2}}\,(-1+2\,\lambda)\\
&-2\,q^{\frac{3}{2}}\,\lambda^{\frac{3}{2}}\,\left(1+\lambda+\lambda^{2}\right)+4\,q^{2}\,\lambda^{2}\,\left(1+\lambda+\lambda^{2}\right)\\
&+q^{5+p}\,\lambda\,\left(-1+\lambda+2\,\lambda^{2}-2\,\lambda^{3}\right)+q^{11+p}\,\lambda\,\left(-1+\lambda+2\,\lambda^{2}-2\,\lambda^{3}\right)\\
&+2\,q^{\frac{15}{2}+p}\,\sqrt{\lambda}\,\left(-1+\lambda+\lambda^{2}+\lambda^{3}\right)-2\,q^{\frac{17}{2}+p}\,\sqrt{\lambda}\,\left(-1+\lambda+\lambda^{2}+\lambda^{3}\right)\\
&+q^{6+p}\,\left(1-\lambda-2\,\lambda^{2}-\lambda^{4}+\lambda^{5}\right)+q^{10+p}\,\left(1-\lambda-2\,\lambda^{2}-\lambda^{4}+\lambda^{5}\right)\\
&+q^{7+p}\,\lambda\,\left(-2+2\,\lambda+6\,\lambda^{2}-2\,\lambda^{3}+\lambda^{4}+\lambda^{5}+\lambda^{6}+\lambda^{7}\right)\\
&+q^{9+p}\,\lambda\,\left(-2+2\,\lambda+6\,\lambda^{2}-2\,\lambda^{3}+\lambda^{4}+\lambda^{5}+\lambda^{6}+\lambda^{7}\right)\\
&-2\,q^{8+p}\,\left(-1+2\,\lambda^{2}+2\,\lambda^{4}+\lambda^{6}+\lambda^{7}+\lambda^{8}\right)\Big)\Big)
\end{aligned}
\tag{B.3}
$$

B.2 Connected sum of trefoil and trefoil with framing p

The composite knot invariants for the connected sum of two knots $\mathcal{K}_1$ and $\mathcal{K}_2$ will be

$$
\mathcal{H}_{(R,S)}[\mathcal{K}_1\#\mathcal{K}_2]=\frac{\mathcal{H}_{(R,S)}[\mathcal{K}_1]\mathcal{H}_{(R,S)}[\mathcal{K}_2]}{\mathcal{H}_{(R,S)}[U]}\ . \tag{B.4}
$$

Similarly, the $SO(N)$ invariants $\mathcal{G}_R[\mathcal{K}_1\#\mathcal{K}_2$ will also be product of $SO(N)$ invariants of the two knots normalised by the $SO(N)$ unknot invariant. From these invariants, we can compute the SO reformulated invariants for the connected sum of two knots. For the connected sum of trefoil, the reformulated polynomial for various representations are:

$$
\begin{aligned}
g_{\square} = {}& (-1)^{p}\,q^{-2}\,\lambda^{2+\frac{p}{2}}\,\Big((-1+\lambda)^{2}+q^{4}\,(-1+\lambda)^{2}+\sqrt{q}\,(-1+\lambda)\,\lambda^{\frac{3}{2}}\\
&-q^{\frac{7}{2}}\,(-1+\lambda)\,\lambda^{\frac{3}{2}}+q^{2}\,\left(2-4\,\lambda+3\,\lambda^{2}\right)-q^{\frac{3}{2}}\,\sqrt{\lambda}\,\left(-1+\lambda^{3}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+q^{\frac{5}{2}}\,\sqrt{\lambda}\,\left(-1+\lambda^3\right)-q\,\left(-2+3\,\lambda-2\,\lambda^2+\lambda^3\right)\\
&-q^3\,\left(-2+3\,\lambda-2\,\lambda^2+\lambda^3\right)\Big)
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
g_{\square\square} = {} & \frac{\lambda^{3+p}}{2\,(-1+q)^2\,q^4\,(1+q)}\,[(-1-q)\\
&\left(-1+\left(\sqrt{q}-\sqrt{\lambda}\right)\left(\sqrt{q}\,(-q+q\lambda-\lambda)-\sqrt{\lambda}\right)\right)^4\left(\sqrt{q}+\sqrt{\lambda}\right)^2\left(-1+\sqrt{q}\sqrt{\lambda}\right)^2\\
&+2\,q\,(1+q)\,(-1+\lambda)^2\left(1+q^2-q\,\lambda\right)^4-2\,q^{1+p}\,(-1+\lambda)\,(-1+q\,\lambda)\\
&\left(1+q^4-q^3\,(-1+\lambda)-q^6\,(-1+\lambda)-q^2\,\lambda+q^5\,(-1+\lambda)\,\lambda\right)^2\\
&+(1+q)\,(q-\lambda)\,(-1+q\,\lambda)\,\left(\left(1+q^2\right)^2-\left(2+q\,(-1+2\,q)\,\left(1+q^2\right)\right)\,\lambda\right.\\
&\left.+(1+(-1+q)\,q)^2\,\lambda^2+(-1+q)^2\,q\,\lambda^3\right)^2\\
&+2\,q^{\frac{1}{2}+p}\,(-1+\lambda)\left(-\sqrt{q}-\sqrt{\lambda}+q^2\,\sqrt{\lambda}+q^{\frac{3}{2}}\,\lambda\right)\\
&\left(-1-q^{\frac{3}{2}}\,\sqrt{\lambda}+q\,\lambda+q^2\,\lambda+q^{\frac{5}{2}}\,\lambda^{\frac{3}{2}}-q^{\frac{17}{2}}\,\lambda^{\frac{3}{2}}+q^{\frac{11}{2}}\,\lambda^{\frac{5}{2}}+q^{\frac{19}{2}}\,\lambda^{\frac{5}{2}}+q^7\,\lambda^3\right.\\
&+q^{\frac{7}{2}}\,\sqrt{\lambda}\,(1+\lambda)-q^9\,\lambda^2\,(1+\lambda)-q^{\frac{9}{2}}\,\sqrt{\lambda}\,(1+\lambda)^2+q^{\frac{13}{2}}\,\sqrt{\lambda}\,(1+\lambda)^2\\
&+q^8\,\lambda\,(1+\lambda)^2+q^5\,\lambda\,(2+\lambda)-q^{\frac{15}{2}}\,\lambda^{\frac{3}{2}}\,(1+2\,\lambda)+q^4\,\left(-1+\lambda+\lambda^2\right)\\
&\left.-q^3\,\left(1+\lambda+\lambda^2\right)-q^6\,\left(1+\lambda+3\,\lambda^2+\lambda^3\right)\right)^2\Big]
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
g_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = {} & \frac{\lambda^{3+p}}{2\,(-1+q)^2\,q^4\,(1+q)}\,[(-1-q)\\
&\left(-1+\left(\sqrt{q}-\sqrt{\lambda}\right)\left(\sqrt{q}\,(-q+q\lambda-\lambda)-\sqrt{\lambda}\right)\right)^4\left(\sqrt{q}+\sqrt{\lambda}\right)^2\left(-1+\sqrt{q}\,\sqrt{\lambda}\right)^2\\
&+2\,q\,(1+q)\,(-1+\lambda)^2\left(1+q^2-q\,\lambda\right)^4+2\,q^{-3-p}\,(q-\lambda)\,(-1+\lambda)\\
&\left(1+q^2+q^6-q^3\,(-1+\lambda)-\lambda-q^4\,\lambda+q\,(-1+\lambda)\,\lambda\right)^2\\
&+(1+q)\,(q-\lambda)\,(-1+q\,\lambda)\,\left(\left(1+q^2\right)^2-\left(2+q\,(-1+2\,q)\,\left(1+q^2\right)\right)\,\lambda\right.\\
&\left.+(1+(-1+q)\,q)^2\,\lambda^2+(-1+q)^2\,q\,\lambda^3\right)^2\\
&+2\,q^{-\left(\frac{21}{2}\right)-p}\,(-1+\lambda)\,\left(-q^{\frac{3}{2}}-\sqrt{\lambda}+q^2\,\sqrt{\lambda}+\sqrt{q}\,\lambda\right)\\
&\left(q^{\frac{19}{2}}-q^8\,\sqrt{\lambda}-q^{\frac{15}{2}}\,\lambda-q^{\frac{17}{2}}\,\lambda-q\,\lambda^{\frac{3}{2}}+q^7\,\lambda^{\frac{3}{2}}+\lambda^{\frac{5}{2}}+q^4\,\lambda^{\frac{5}{2}}-q^{\frac{5}{2}}\,\lambda^3\right.\\
&+q^6\,\sqrt{\lambda}\,(1+\lambda)+\sqrt{q}\,\lambda^2\,(1+\lambda)+q^3\,\sqrt{\lambda}\,(1+\lambda)^2-q^5\,\sqrt{\lambda}\,(1+\lambda)^2\\
&-q^{\frac{3}{2}}\,\lambda\,(1+\lambda)^2-q^{\frac{9}{2}}\,\lambda\,(2+\lambda)-q^2\,\lambda^{\frac{3}{2}}\,(1+2\,\lambda)-q^{\frac{11}{2}}\,\left(-1+\lambda+\lambda^2\right)\\
&\left.+q^{\frac{13}{2}}\,\left(1+\lambda+\lambda^2\right)+q^{\frac{7}{2}}\,\left(1+\lambda+3\,\lambda^2+\lambda^3\right)\right)^2\Big]
\end{aligned}
\tag{B.7}
$$

B.3 Hopf link with framing p_1 on the first strand and p_2 on the second

$$g_{\square,\square} = \frac{(-1)^{p_1+p_2}\lambda^p\;(-1+q)\;(-1+\lambda)}{\sqrt{q}\,\sqrt{\lambda}} \tag{B.8}$$

$$g_{\square\square\,,\square} = \frac{(-1)^{p_2}\;(-1+\lambda)\;\left(-q^{\frac{1}{2}+p_1} - 2\,q^{\frac{3}{2}}\;(-1+\lambda) + q^{\frac{3}{2}+p_1}\;(-1+\lambda) + q^{\frac{5}{2}+p_1}\,\lambda\right)}{q^{\frac{3}{2}}\,\lambda} \tag{B.9}$$

$$g_{\begin{smallmatrix}\square\\\square\end{smallmatrix},\square} = -\left(\frac{(-1)^{p_2}\,q^{-1-p_1}\;(-1+\lambda)\;(q+q^2+2\,q^{1+p_1}\;(-1+\lambda)-\lambda-q\,\lambda)}{\lambda}\right) \tag{B.10}$$

These reformulated polynomials obey Marino's conjecture (3.6).

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